For problems 1 through 3, consider the discrete dynamical system $\vec{x}_{k+1} = A \vec{x}_k$.

1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.2 & 0 \\ 0 & 0.8 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1.2 & -0.4 \\ 0 & 0.8 \end{bmatrix}$.

   (a) Calculate and plot $\vec{x}_0, \vec{x}_1, \vec{x}_2,$ and $\vec{x}_3$ (i.e. the first four terms of the trajectory) for $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (b) Calculate and plot $\vec{x}_0, \vec{x}_1, \vec{x}_2,$ and $\vec{x}_3$ for $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   (c) Calculate and plot $\vec{x}_0, \vec{x}_1, \vec{x}_2,$ and $\vec{x}_3$ for $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (d) Find the eigenvalues of $A$, and classify the origin as an attractor, repeller, saddle point, spiral attractor, spiral repeller, orbital center, or none of these for this dynamical system.

2. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & \sqrt{3} \end{bmatrix}$.

   (a) Calculate and plot $\vec{x}_0, \vec{x}_1, \vec{x}_2,$ and $\vec{x}_3$ for $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

   (b) Calculate and plot $\vec{x}_0, \vec{x}_1, \vec{x}_2,$ and $\vec{x}_3$ for $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (c) Find the characteristic polynomial and eigenvalues of $A$, and classify the origin as an attractor, repeller, saddle point, spiral attractor, spiral repeller, orbital center, or none of these for this dynamical system.

   FYI: The matrix $A$ for problem 2 does not have a diagonalization over $\mathbb{R}$. However it does have a diagonalization over $\mathbb{C}$ given by $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} + i & 0 \\ 0 & \sqrt{3} - i \end{bmatrix} \begin{bmatrix} i & -i \end{bmatrix}$.

3. For each of the following choices of $A$, classify the origin as an attractor, repeller, saddle point, spiral attractor, spiral repeller, orbital center, or none of these for the associated dynamical system.

   a) $\begin{bmatrix} 0.1 & 0.6 \\ 2 & -0.1 \end{bmatrix}$

   b) $\begin{bmatrix} 0.4 & -0.95 \\ 0.8 & 0.6 \end{bmatrix}$

   c) $\begin{bmatrix} \frac{3}{4} & 5 \\ \frac{3}{4} & \frac{7}{2} \end{bmatrix}$

   d) $\begin{bmatrix} 1 & 1.6 \\ 0.5 & 1.2 \end{bmatrix}$
4. The residents of the eccentric town of Fair Weather are renowned for their shifting sports loyalties. Initially one third of the population rooted for Penn State, one third rooted for Michigan, and one third rooted for Ohio State. However after every week, 30% of the Penn State fans changed their loyalty to Michigan and 10% changed their loyalty to Ohio State, 20% of Michigan fans switched loyalty to Penn State and 15% changed loyalty to Ohio State, and 10% of Ohio State fans switched to Michigan and 20% switched to Penn State.

   a) Set up a transition matrix for this Markov process, using the state vector \( \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) where \( x \) corresponds to the fraction of Penn State fans, \( y \) corresponds to the fraction of Michigan fans, and \( z \) corresponds to the fraction of Ohio State fans.

   b) What percentage of Fair Weather fans were rooting for Ohio State after one week? after two weeks? after three weeks?

   c) Find the steady state probability vector of this Markov process.

5. Ampere Rent-a-Car has several offices for pick-up and return of vehicles, one at the airport and several around town. Each week, 95% of cars at the airport are unrented or are rented and returned to the airport while 5% are returned in town. And each week, 90% of cars in town are unrented or are rented and returned in town while 10% are returned at the airport.

   a) Set up a transition matrix \( P \) for this Markov process. Declare the state vector \( \vec{x} \) that you are using with this transition vector, i.e. indicate what you are using each entry of \( \vec{x} \) to represent. (Note: The latter step could also be called “documenting your variables”.)

   b) Suppose that Ampere currently has 60% of its rental cars at its town offices and 40% at the airport. What percentage of cars does it have at the airport and what percentage does it have in town after one week? after two weeks?

   c) Find the steady state probability vector of this Markov process.

   d) What are the eigenvalues of \( P \)?

   e) (for ‘e’xtra thinking) What is the significance of the eigenvalues of this particular \( P \) in this context? [Note: The significance of one of the eigenvalues should be reasonably clear to discern. But it may take more contemplation to say something cogent about the other.]