EXTENDED SHOCKWAVE DECOMPOSABILITY RELATED TO BOUNDARIES OF HOLOMORPHIC 1-CHAINS WITHIN $\mathbb{CP}^2$

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Abstract. We consider the notion of meromorphic Whitney multifunction solutions to $ff_\xi = f_\eta$, which yields an enhanced version of the Dolbeault Henkin characterization of boundaries of holomorphic 1-chains within $\mathbb{CP}^2$. By analyzing the equations describing meromorphic Whitney multifunction solutions to $ff_\xi = f_\eta$ and by creating some generalizations of certain linear dependence results, we show that a function G may be decomposed into a sum of such solutions, modulo ξ-affine functions and with a selected bound on the degree of such sum, if and only if $G_{\xi\xi}$ satisfies a finite set of explicitly constructible partial differential equations.

1. Introduction

Dolbeault and Henkin introduced a characterization of boundaries of holomorphic 1-chains within $\mathbb{CP}^n$, with a subsequent expansion to a characterization of boundaries of holomorphic p-chains within $\mathbb{CP}^n$ [4], [5]. At the heart of their general result, both in proof and in essence, is the case of boundaries of holomorphic 1-chains within $\mathbb{CP}^2$.

Roughly speaking, the Dolbeault Henkin characterization is expressed in terms of the "holomorphic shockwave decomposability" of a particular integral function. Specifically, a closed, oriented, $\mathcal{C}^2$ 1-chain γ contained in $\mathbb{C}^2 \subset \mathbb{CP}^2$ bounds a holomorphic 1-chain within $\mathbb{CP}^2$ if and only if there exists some point (ξ^*, η^*) in $\mathfrak{U}_\gamma := \{(\xi, \eta) \mid \operatorname{spt} \gamma \cap \{z_2 = \xi + \eta z_1\} = \emptyset\}$ about which the function $G_\gamma(\xi, \eta) := \frac{1}{2\pi \mathrm{i}} \int_\gamma z_1 \frac{d(z_2 - \eta z_1)}{z_2 - \xi - \eta z_1}$ can be locally decomposed, modulo ξ-affine functions, into a $\mathbb{Z}$-linear combination of germs of holomorphic solutions to the partial differential equation $f_\eta = ff_\xi$ [4], [5]. Work by Dinh extends this result to the case where γ is a rectifiable 1-current whose support satisfies a condition called A_1 [3].

In the proof of the above, one discovers that a multiset of germs of holomorphic solutions to $f_\eta = ff_\xi$ at (ξ^*, η^*) can be used to encode a local portion of a

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[0] *Math Subject Classifications.* 32C30, 35Q53, 35N10

[0] *Key Words and Phrases.* Boundaries of holomorphic chains, Burgers equation, Sums of shockwaves, Overdetermined systems

[0] *Acknowledgments and Notes.* This work was partially supported by a NSF Graduate Fellowship while the author was at the University of Michigan.

generic positive holomorphic 1-chain near the line $z_2 = \xi^* + \eta^* z_1$. However, positive holomorphic 1-chains with local components that intersect the line $z_2 = \xi^* + \eta^* z_1$ non-transversally or at infinity are examples that cannot be encoded by this approach. If γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$, then G_γ is holomorphic shockwave decomposable, in the sense of the above paragraph, about (ξ^*, η^*) for a generic point (ξ^*, η^*) in $\mathfrak{U}_\gamma$, but not for every (ξ^*, η^*) in $\mathfrak{U}_\gamma$ in general. So, with the above type of shockwave decomposability, one must either concede to permitting genericity in the choice of the point (ξ^*, η^*) or to only detecting holomorphic 1-chains bounded by γ that satisfy certain generic restrictions near $z_2 = \xi^* + \eta^* z_1$. (Such restrictions are generic among the collection of all holomorphic 1-chains. However with (ξ^*, η^*) fixed, one can readily construct examples of γ that bound holomorphic 1-chains, but none of which satisfy such a restriction.)

It would be ideal to have full freedom to fix the point (ξ^*, η^*) in $\mathfrak{U}_\gamma$ while allowing general holomorphic 1-chain behavior near the line $z_2 = \xi^* + \eta^* z_1$. We may accomplish this by using meromorphic Whitney multifunction solutions to $f_\eta = f f_\xi$ instead of unramified holomorphic solutions. As we will show, meromorphic Whitney multifunction solutions to $f_\eta = f f_\xi$ can be represented as the roots to a ζ-polynomial $P_0(\xi,\eta)\zeta^N - P_1(\xi,\eta)\zeta^{n-1} + \cdots + (-1)^N P_N(\xi,\eta)$ where $P_0, P_1, \ldots, P_N$ satisfy the *refined h.s.w. equations*

$$P_0[(P_{k+1})_\xi + (P_k)_\eta] = P_k[(P_1)_\xi + (P_0)_\eta], \text{ for } 1 \le k \le N, \quad \text{and} \quad (P_0)_\xi = 0, \tag{1.1}$$

with the occurrence of P_{N+1} treated as 0. Also, one can prescribe a canonical way to choose the functions $P_0, P_1, \ldots, P_N$, which we show at the end of Section 4.

Related to this, for $N \ge 0$ we say that $\mu(\xi,\eta)$ satisfies condition $(\star_N)$ if there exist $P_0, P_1, \ldots, P_N$ satisfying

$$(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \text{ for } 0 \le k \le N, \quad \text{and} \quad (P_0)_\xi = 0. \tag{1.2}$$

with P_{N+1} is regarded as zero. (Notably, this implies that $\mu_\xi = D_\xi^2\left(\frac{P_1}{P_0}\right)$.) The Dolbeault Henkin characterization within $\mathbb{CP}^2$ can be adapted as follows.

Theorem 1.1. *Let γ be a closed, rectifiable 1-current whose support is contained in $\mathbb{C}^2 \subset \mathbb{CP}^2$ and satisfies condition A_1. Let $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$. Then γ bounds a holomorphic 1-chain with finite mass within $\mathbb{CP}^2$ if and only if there exist closed, rectifiable 1-currents γ^+ and γ^-, with no identically oriented arc common to both, such that $\gamma = \gamma^+ - \gamma^-$ with $\mu^+(\xi,\eta) := D_\xi G_{\gamma^+}(\xi,\eta)$ and $\mu^-(\xi,\eta) := D_\xi G_{\gamma^-}(\xi,\eta)$ satisfying condition $(\star_{N^+})$ and $(\star_{N^-})$, respectively, in a neighborhood of (ξ^*, η^*), for some non-negative integers N^+ and N^-.*

Remark: There exist γ^+ and γ^- satisfying this theorem for a fixed N^+ and N^- if and only if γ bounds a holomorphic 1-chain with finite mass within $\mathbb{CP}^2$

that has at most N^+ positive intersections and N^- negative intersections, counting multiplicity, with the line $z_2 = \xi^* + \eta^* z_1$.

It is natural to inquire about the practicality of determining when G_γ is shockwave decomposable. Towards this end, we examine condition $(\star_N)$. (The equations in (1.2) have some intriguing features that may draw independent interest from the vantage point of integrable systems.) As may be seen, μ satisfies condition $(\star_N)$ if and only there exists a solution to a particular overdetermined system of partial differential equations involving μ. We show that this is equivalent to μ_ξ satisfying a particular set of partial differential equations that depends on N. This is expressed in Theorem 6.7, a special case of which is given below. (The following employs some definitions that will be given in Section 3 and Section 6. For now we simply remark that $W_Y^{T_N,\check{U}_N}(M_N)$ is an modified form of a generalized Wronskian, indexed by Young diagrams of size $\binom{N+2}{2}$ denoted by Y, and that it is purely a partial differential expression of μ_ξ.)

Theorem 1.2. *Let μ be a germ of a holomorphic function about a point (ξ^*, η^*), and let N be a non-negative integer. The function μ satisfies condition $(\star_N)$ if and only if $W_Y^{T_N,\check{U}_N}(M_N) = 0$ for each Young diagram Y of cardinality $\binom{N+2}{2}$.*

We briefly note some consequences of these results. For one, consider the collection of γ that bound a holomorphic 1-chain V such that V intersects the line $z_2 = \xi^* + \eta^* z_1$ only positively and with total degree at most N. As a consequence of the previous theorems, this collection of γ can be characterized by a finite set of explicit partial differential equations on $D_\xi^2 G_\gamma$.

Also, if γ is a closed, finite 1-chain γ with finitely many self-intersections, then there are only finitely many potential ways that γ can be decomposed into $\gamma^+ - \gamma^-$ with γ^+ and γ^- having no comman arcs. Let $\{\gamma_j\}$ be a finite family of subcurves of γ that generates all of the simple closed curves in γ. So we can characterize whether such a γ bounds a holomorphic 1-chain V, with separately prescribed bounds on the degree of positive and negative intersections between V and $z_2 = \xi^* + \eta^* z_1$, using a finite number of partial differential equations on $\{D_\xi^2 G_{\gamma_j}\}$.

In Section 2 we present some preliminaries and notation. In Section 3 we generalize a result originally regarding linear dependence into a broader differential equation context. We introduce holomorphic and meromorphic Whitney multifunction solutions to the shockwave equation $f_\eta = f f_\xi$ in Section 4, deriving the relevant formulae there. In Section 5, we establish the ensuing extension of the Dolbeault Henkin characterization of boundaries of holomorphic 1-chains within $\mathbb{CP}^2$. We examine our extended notion of shockwave decomposability, yielding proofs of Theorem 1.1 and Theorem 1.2, in Section 6 with some details relegated to the appendix.

2. Preliminaries and Auxiliary Results

2.1. **Some Relevant Function Algebras and Definitions.** Let ${}_m\mathcal{O}$ denote the sheaf of germs of holomorphic functions on $\mathbb{C}^m$ and let ${}_m\mathcal{M}$ denote the sheaf of germs of meromorphic functions on $\mathbb{C}^m$. For our purposes, it makes sense to regard $\mathbb{C}^0$ as a single isolated point and to treat ${}_0\mathcal{O}$ and ${}_0\mathcal{M}$ as the sheaf of complex constants. For the time being we fix $m \geq 1$ and omit the pre-subscript denotation.

For a domain Ω in $\mathbb{C}^m$, $\mathcal{O}(\Omega) = \Gamma(\Omega, \mathcal{O})$ denotes the ring of holomorphic functions on Ω. And for a non-empty, connected, compact set K in $\mathbb{C}^m$, $\mathcal{O}(K) = \Gamma(K, \mathcal{O})$ denotes the ring of germs of holomorphic functions on K. The field of meromorphic functions on Ω and the field of germs of meromorphic functions on K are denoted $\mathcal{M}(\Omega)$ $(= \Gamma(\Omega, \mathcal{M}))$ and $\mathcal{M}(K)$ $(= \Gamma(K, \mathcal{M}))$, respectively. On the local level, it holds by definition that $\mathcal{M}_p$ equals the fraction field of $\mathcal{O}_p$, i.e. $\mathsf{ff}(\mathcal{O}_p)$. If Ω, resp. K, is Stein, then this algebraic statement also holds in a more global sense, namely $\mathcal{M}(\Omega) = \mathsf{ff}(\mathcal{O}(\Omega))$, resp. $\mathcal{M}(K) = \mathsf{ff}(\mathcal{O}(K))$ [15](Theorem 8.19).

For a point $p \in \mathbb{C}^m$, the ring $\mathcal{O}_p$ is a Noetherian, unique factorization domain [9]. It also holds that $\mathcal{O}(K)$ is a Noetherian, unique factorization domain for a broad range of compact sets K [7], [17], [2]. For instance, it is sufficient to let K be a compact, semi-analytic, Stein set in $\mathbb{C}^m$ such that $H^2(K;\mathbb{Z}) = 0$; e.g. a closed polydisk will do.

Let $\mathcal{O}' = {}_{m-1}\mathcal{O}$ and $\mathcal{M}' = {}_{m-1}\mathcal{M}$. The sheaves $\mathcal{O}'$ and $\mathcal{M}'$ have natural, respective inclusions in $\mathcal{O}$ and $\mathcal{M}$. So we may define the sheaf $\mathcal{M}'\mathcal{O}$, which is equivalent to the localization $(\mathcal{O}'\backslash\{0\})^{-1}\mathcal{O}$.

Let $(\vec{\eta}, \xi) = (\eta_1, \eta_2, \ldots, \eta_{m-1}, \eta_m = \xi)$ be the coordinates for $\mathbb{C}^m$, and let $(\vec{\eta}^*, \xi^*)$ be a fixed point in $\mathbb{C}^m$. We say that a non-empty, compact set K is *Cauchy-viable* with respect to $\xi = \xi^*$ if each non-empty slice of the form $K \cap \{\vec{\eta} = \vec{\eta}_0\}$, for $\vec{\eta}_0 \in \mathbb{C}^{m-1}$, contains the point $(\vec{\eta}_0, \xi^*)$ and possesses a neighborhood basis of simply connected domains. (If $m = 1$, then we simply mean that K contains ξ^* and has a neighborhood basis of simply connected domains.)

Assume that K is a non-empty, connected, Cauchy-viable, Stein, compact set, and let $\tilde{K} = \{\vec{\eta} \in \mathbb{C}^{m-1} \mid (\vec{\eta}, \xi^*) \in K\}$. Under this assumption, $\mathcal{O}'(\tilde{K})$ and $\mathcal{M}'(\tilde{K})$ have natural, respective inclusions in $\mathcal{O}(K)$ and $\mathcal{M}(K)$. Also it holds that $\mathcal{M}'\mathcal{O}(K)$ $(= \Gamma(K, \mathcal{M}'\mathcal{O}))$ can be identified with $\{g/h \in \mathcal{M}(K) \mid g \in \mathcal{O}(K), h \in \mathcal{O}'(\tilde{K})\backslash\{0\}\}$, based on arguments such as Theorem 8.19 and Lemma 8.5 of [15].

The following observations motivate the previous definitions and will be useful later on. For $f = g/h \in \mathcal{M}'\mathcal{O}(K)$, with $g \in \mathcal{O}(K)$ and $h \in \mathcal{O}'(\tilde{K})\backslash\{0\}$, $\int_{\xi^*}^{\xi} f(\vec{\eta}, \xi')\, d\xi' = (\int_{\xi^*}^{\xi} g(\vec{\eta}, \xi')\, d\xi')/h(\vec{\eta})$ yields a well-defined element of $\mathcal{M}'\mathcal{O}(K)$. (This integral is not so well-behaved on general elements of $\mathcal{M}(K)$, as one may see with examples such as $1/\xi$.) Also if $u \in \mathcal{M}(K)$ and $D_\xi u = f$ for some $f \in \mathcal{M}'\mathcal{O}(K)$, then $\tilde{u}(\vec{\eta}) := u(\vec{\eta}, \xi^*)$ defines an element of $\mathcal{M}'(\tilde{K})$, i.e. $\tilde{\mu} \not\equiv \infty$, and $u \in \mathcal{M}'\mathcal{O}(K)$. Furthermore, for any $f \in \mathcal{M}'\mathcal{O}(K)$ and $g \in \mathcal{M}'(\tilde{K})$, there is a unique solution $u \in \mathcal{M}'\mathcal{O}(K)$ to the initial value problem $D_\xi u = f$ and $u|_{\xi=\xi^*} = g$.

2.2. **Matrix and Indexing Protocol.** If M is a matrix with rows indexed by a finite ordered set A and columns indexed by a finite ordered set B, we call M a $A \times B$ matrix. For $\alpha \in A$ and $\beta \in B$, we use the notation M_α^β to refer to the entry row-referenced by α and column-referenced by β. (For matrices with one row or one column, we may drop the trivial index from the notation.) For a $A \times B$ matrix M and a $B \times C$ matrix L, the matrix product ML satisfies the equation $(ML)_\alpha^\gamma = \sum_{\beta \in B} M_\alpha^\beta L_\beta^\gamma$, for $\alpha \in A$, $\gamma \in C$. For $A \times A$ matrices, the notions of triangularity and strict triangularity can be defined. We say that M is lower (resp. upper) triangular if $M_{\alpha_1}^{\alpha_2} = 0$ whenever $\alpha_1 \prec \alpha_2$ (resp. $\alpha_2 \prec \alpha_1$), and we say that M is strictly lower (resp. upper) triangular if $M_{\alpha_1}^{\alpha_2} = 0$ whenever $\alpha_1 \preceq \alpha_2$ (resp. $\alpha_2 \preceq \alpha_1$).

3. A Generalization of Certain Results on Linear Dependence

In this section, our interest centers on the space of the type

$$\ker \phi \cap \bigcap_{j=1}^{m} \ker(D_{\eta_j} - A_j), \tag{3.1}$$

where ϕ is a row matrix and A_j are square matrices, both with entries being functions of $\eta_1, \ldots, \eta_m$, subject to certain requirements and relationships. In other words, this is the space of mutual solutions to a linear equation and certain systems of ordinary differential equations. Among other things, we develop a means for determining when such a space is non-trivial.

While the results in this section are much more general, they are specifically motivated by the space $\ker M_N \cap \ker(D_\xi - A_N) \cap \ker(D_\eta - B_N)$ that is introduced in Subsection 6.1. Also, this section is a generalization of the results and techniques presented in an article on linear dependence [19]. Note that the space of linear relations among entries of a row matrix ϕ is simply $\ker \phi \cap \bigcap_{j=1}^m \ker D_{\eta_j}$, which corresponds to the special case where each A_j is zero in (3.1).

Let $(\vec{\eta}, \xi) = (\eta_1, \ldots, \eta_{m-1}, \eta_m = \xi)$ denote coordinates for $\mathbb{C}^m$. For the first portion of this subsection, specifically Lemma 3.1 through Lemma 3.4, we assume that K is a non-empty, connected, Stein, compact set in $\mathbb{C}^m$ and that K is Cauchy-viable with respect to $\xi = \xi^*$. Let $\tilde{K} = \{\vec{\eta} \in \mathbb{C}^{m-1} \mid (\vec{\eta}, \xi^*) \in K\}$.

Let A be a $N \times N$ matrix with entries in $\mathcal{M}'\mathcal{O}(K)$. Define T to be the operator acting on $1 \times N$ matrices with entries in $\mathcal{M}(K)$ such that $T(\psi) = D_\xi(\psi) + \psi A$. For $f \in \mathcal{M}(K)$, T satisfies the Leibniz-like identity

$$T(f\psi) = f_\xi \psi + fT(\psi). \tag{3.2}$$

Also, for a column vector $\vec{v}$ in $\mathcal{M}(K)^N$, it holds that

$$D_\xi(\psi\vec{v}) = T(\psi)\vec{v} + \psi(D_\xi - A)\vec{v}. \tag{3.3}$$

Let $\{\phi_\alpha\}$ denote some (possibly infinite) collection of $1 \times N$ matrices with entries in $\mathcal{M}'\mathcal{O}(K)$. Let $\mathcal{N}(\psi) = \cap_{j=0}^\infty \ker T^j(\psi) \subseteq \mathcal{M}(K)^N$, and let $\mathcal{N} = \cap_\alpha \mathcal{N}(\phi_\alpha)$. Using (3.3), we see that $\ker\psi \cap \ker(D_\xi - A) \subseteq \ker T(\psi)$, and so

$$\cap_\alpha \ker\phi_\alpha \cap \ker(D_\xi - A) = \mathcal{N} \cap \ker(D_\xi - A). \tag{3.4}$$

Now suppose that $A = S^{-1}LS$ for some $N \times N$ matrices S and L such that $S \in \mathrm{GL}_N(\mathcal{M}'(\tilde{K}))$ and L is strictly lower triangular with entries in $\mathcal{M}'\mathcal{O}(K)$. One useful property of $\mathcal{N}$ is the following.

Lemma 3.1.

$$\mathcal{N} = (\mathcal{N} \cap \ker(D_\xi - A)) \otimes_{\mathcal{M}'(\tilde{K})} \mathcal{M}(K). \tag{3.5}$$

Proof. By a change of variable it is sufficient to consider the case $A = L$. The right-hand side of (3.5) is tautologically contained in $\mathcal{N}$. For the reverse inclusion, it suffices to show that $\mathcal{N}$ has a $\mathcal{M}(K)$ basis contained in $\ker(D_\xi - A)$.

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a $\mathcal{M}(K)$ basis for $\mathcal{N}$. By reducing this basis "from the top" we may assume that there exist $\ell_1 < \ell_2 < \cdots < \ell_k$ such that for each j, $(\vec{v}_j)_{\ell_j} = 1$ and $(\vec{v}_j)_\ell = 0$ for $\ell < \ell_j$.

Note that $(D_\xi - A)$ maps $\mathcal{N}$ to itself, as (3.3) shows that $T^j(\phi_\alpha)(D_\xi - A)\vec{v} = D_\xi(T^j(\phi_\alpha)\vec{v}) - T^{j+1}(\phi_\alpha)\vec{v}$. By the strict lower triangularity of A and the reduced form of the basis, it holds that $(D_\xi - A)\vec{v}_j \in \mathrm{span}_{\mathcal{M}(K)}\{\vec{v}_{j+1}, \dots, \vec{v}_k\}$. In particular, $(D_\xi - A)\vec{v}_k = 0$. So there exists a $j \le k-1$ and a $t \ge j+1$ such that $(D_\xi - A)\vec{v}_{j'} = 0$ for $j+1 \le j' \le k$ and $(D_\xi - A)\vec{v}_j \in \mathrm{span}_{\mathcal{M}(K)}\{\vec{v}_t, \dots, \vec{v}_k\}$.

As $(\vec{v}_j)_{\ell_j} = 1$ and $D_\xi(\vec{v}_j)_\ell = \sum_{i=\ell_j}^{\ell-1} A_\ell^i (\vec{v}_j)_i$ for $\ell_j < \ell < \ell_t$, we may inductively conclude that $(\vec{v}_j)_\ell \in \mathcal{M}'\mathcal{O}(K)$ for $\ell < \ell_t$. Thus there exists a $\lambda \in \mathcal{M}'\mathcal{O}(K)$ such that $\lambda_\xi = \sum_{i=\ell_j}^{\ell_t-1} A_{\ell_t}^i (\vec{v}_j)_i$. Let $\vec{w} = \vec{v}_j + (\lambda - (\vec{v}_j)_{\ell_t})\vec{v}_t$, and observe that $((D_\xi - A)\vec{w})_{\ell_t} = 0$. In replacing $\vec{v}_j$ with $\vec{w}$, we preserve $\{\vec{v}_1, \dots, \vec{v}_k\}$ as a $\mathcal{M}(K)$ basis of $\mathcal{N}$, yet $(D_\xi - A)\vec{v}_j \in \mathrm{span}_{\mathcal{M}(K)}\{\vec{v}_{t+1}, \dots, \vec{v}_k\}$ (or $(D_\xi - A)\vec{v}_j = 0$ if $t = k$). By induction in t and j, we produce a $\mathcal{M}(K)$ basis for $\mathcal{N}$ contained in $\ker(D_\xi - A)$. □

Let R be the fundamental matrix of the vector differential equation $(D_\xi - A)\vec{v} = 0$ normalized at $\xi = \xi^*$, i.e. R is the unique solution to the matrix Cauchy problem $R_\xi = AR$ and $R|_{\xi=\xi^*} = \mathrm{Id}$ [1](pp. 1-2). So, for a column vector $\vec{w}$ in $\mathcal{M}'(\tilde{K})^N$, $\vec{v} = R\vec{w}$ gives the unique solution to the vector differential equation $(D_\xi - A)\vec{v} = 0$ with Cauchy data $\vec{v}|_{\xi=\xi^*} = \vec{w}$. Thus $\vec{w} \mapsto R\vec{w}$ is a $\mathcal{M}'(\tilde{K})$ isomorphism between $\mathcal{M}'(\tilde{K})^N$ and $\ker(D_\xi - A) \subseteq \mathcal{M}(K)^N$ with the inverse being the map $\vec{v} \mapsto \vec{v}|_{\xi=\xi^*}$.

Lemma 3.2.

(3.6)
$$R\left(\bigcap_{\alpha}\bigcap_{j\geq 0}\ker\left((T^j(\phi_\alpha))|_{\xi=\xi^*}\right)\cap\mathcal{M}'(\tilde{K})^N\right)=\bigcap_{\alpha}\bigcap_{j\geq 0}\ker(T^j(\phi_\alpha))\cap\ker(D_\xi-A)$$

Proof. Let $\vec{w}\in\mathcal{M}'(\tilde{K})^N$. Since $\left(T^j(\phi_\alpha)R\vec{w}\right)\big|_{\xi=\xi^*}=\left(T^j(\phi_\alpha)\right)\big|_{\xi=\xi^*}\vec{w}$ and $D_\xi(T^j(\phi_\alpha)R\vec{w})=T^{j+1}(\phi_\alpha)R\vec{w}$, it follows that $\left(D_\xi^k\left(T^j(\phi_\alpha)R\vec{w}\right)\right)\Big|_{\xi=\xi^*}=\left(T^{j+k}(\phi_\alpha)\right)\big|_{\xi=\xi^*}\vec{w}$ for $j,k\geq 0$. From this, the forward inclusion follows.

The reverse inclusion follows by simply employing the inverse of R and by using the equation $\left(T^j(\phi_\alpha)\right)\big|_{\xi=\xi^*}\vec{w}=\left(T^j(\phi_\alpha)R\vec{w}\right)\big|_{\xi=\xi^*}$.

□

Let B be a $N\times N$ matrix with entries in $\mathcal{M}'\mathcal{O}(K)$, and let $\tilde{B}=B|_{\xi=\xi^*}$. Define the operator U by $U(\psi)=D_{\eta_{m-1}}\psi+\psi B$, which satisfies properties analogous to (3.2) and (3.3). Also suppose that there exist indices $\alpha_1,\ldots,\alpha_q$ and $N\times 1$ matrices $Q_1,\ldots,Q_q$ with entries in $\mathcal{M}(K)$ such that

$$[D_\xi-A,D_{\eta_{m-1}}-B]=\sum_{i=1}^{q}Q_i\phi_{\alpha_i}. \tag{3.7}$$

Lemma 3.3.

$$R\left(\bigcap_{\alpha}\bigcap_{j\geq 0}\ker\left(\left(T^j(\phi_\alpha)\right)\big|_{\xi=\xi^*}\right)\cap\ker(D_{\eta_{m-1}}-\tilde{B})\cap(\mathcal{M}'(\tilde{K}))^N\right)$$
$$=\bigcap_{\alpha}\bigcap_{j\geq 0}\ker T^j(\phi_\alpha)\cap\ker(D_{\eta_{m-1}}-B)\cap\ker(D_\xi-A) \tag{3.8}$$

Proof. Let $\vec{w}\in\mathcal{M}'(\tilde{K})^N$. By Lemma 3.2 and since $(D_{\eta_{m-1}}-\tilde{B})\vec{w}=\left((D_{\eta_{m-1}}-B)R\vec{w}\right)\big|_{\xi=\xi^*}$, it follows that the right-hand side is contained in the left-hand side.

To show the forward inclusion, let $R\vec{w}$ be a member of the left-hand side. From Lemma 3.2, it follows that $\phi_\alpha R\vec{w}=0$ for all α. Thus

$$(D_\xi-A)(D_{\eta_{m-1}}-B)R\vec{w}=(D_{\eta_{m-1}}-B)(D_\xi-A)R\vec{w}+\sum_{i=1}^{q}Q_i\phi_{\alpha_i}R\vec{w}=0. \tag{3.9}$$

Since $(D_{\eta_{m-1}}-B)R\vec{w}\in\ker(D_\xi-A)$, we may use the inverse of R to see that

$$(D_{\eta_{m-1}}-B)R\vec{w}=R\left((D_{\eta_{m-1}}-B)R\vec{w}\right)\big|_{\xi=\xi^*}=R(D_{\eta_{m-1}}-\tilde{B})\vec{w}=0. \tag{3.10}$$

□

For a $N\times 1$ matrix ψ with entries in $\mathcal{M}(K)$, one may easily calculate that

$$U(T(\psi))-T(U(\psi))=\psi[D_\xi-A,D_{\eta_{m-1}}-B]=\sum_{i=1}^{q}(\psi Q_i)\phi_{\alpha_i}. \tag{3.11}$$

A generalization of this is the following.

Lemma 3.4. *Let ψ be a $N \times 1$ matrix ψ with entries in $\mathcal{M}(K)$, and let $j, k \geq 1$. $U^k(T^j(\psi)) - T^j(U^k(\psi))$ is contained in the $\mathcal{M}(K)$ span of terms of the form $T^{j'}(U^{k'}(\phi_{\alpha_i}))$ (or $U^{k'}(T^{j'}(\phi_{\alpha_i}))$) with $1 \leq i \leq q$, $0 \leq j' \leq j-1$, and $0 \leq k' \leq k-1$.*

Proof. Equation (3.11) gives the lemma in the case $j = k = 1$.

Assume that the lemma holds for $1 \leq j \leq j_0$ and $1 \leq k \leq k_0$ for some $j_0, k_0 \geq 1$. Note that

$$(3.12) \quad \begin{aligned} &U^{k_0}(T^{j_0+1}(\psi)) - T^{j_0+1}(U^{k_0}(\psi)) \\ &\qquad = (U^{k_0}T - TU^{k_0})(T^{j_0}(\psi)) + T((U^{k_0}T^{j_0} - T^{j_0}U^{k_0})(\psi)), \end{aligned}$$

and

$$(3.13) \quad \begin{aligned} &U^{k_0+1}(T^{j_0}(\psi)) - T^{j_0}(U^{k_0+1}(\psi)) \\ &\qquad = (UT^{j_0} - T^{j_0}U)(U^{k_0}(\psi)) + U((U^{k_0}T^{j_0} - T^{j_0}U^{k_0})(\psi)), \end{aligned}$$

Consider the right-hand side of (3.12). By the induction hypothesis, we see that the first term is in the $\mathcal{M}(K)$ span of terms of the form $U^{k'}(\phi_{\alpha_i})$ for $0 \leq k' \leq k-1$, and, by also using (3.2), we see that the second term is in the $\mathcal{M}(K)$ span of terms of the form $T^{j'}(U^{k'}(\phi_\alpha))$ for $0 \leq j' \leq j_0$ and $0 \leq k' \leq k_0 - 1$.

Now consider the right-hand side of (3.13). Similar to before, the first term is in the $\mathcal{M}(K)$ span of terms of the form $T^{j'}(\phi_{\alpha_i})$ for $0 \leq j' \leq j_0 - 1$. The second term, by way of (3.2) and a recursive, more protracted application of the induction hypothesis, is in the $\mathcal{M}(K)$ span of terms of the form $T^{j'}(U^{k'}(\phi_{\alpha_i}))$ for $0 \leq j' \leq j_0 - 1$ and $0 \leq k' \leq k_0$.

In order to use the terms $U^{k'}(T^{j'}(\phi_{\alpha_i}))$ in place of $T^{j'}(U^{k'}(\phi_{\alpha_i}))$, simply apply the established portion of the lemma with T and U interchanged.

□

For the remainder of this section, suppose that K is a compact Stein set containing $(\eta_1^*, \eta_2^*, \ldots, \eta_m^*)$ such that $K_j := \{(\eta_1, \ldots, \eta_j) \in \mathbb{C}^j \mid (\eta_1, \ldots, \eta_j, \eta_{j+1}^*, \ldots, \eta_m^*) \in K\}$ is Cauchy-viable with respect to $\eta_j = \eta_j^*$ for $1 \leq j \leq m$.

For $0 \leq j \leq m-1$, define E_j to be restriction by $\eta_{j+1} = \eta_{j+1}^*, \ldots, \eta_m = \eta_m^*$. (Define E_m to be the identity operator.) Thus E_j is a well-defined map from ${}_j\mathcal{M}\,{}_m\mathcal{O}(K)$ to ${}_j\mathcal{M}(K_j)$. We administer the action of E_j entry-wise when it is applied to vectors or matrices.

For $1 \leq j \leq m$, let A_j be a $N \times N$ matrix with entries in ${}_{j-1}\mathcal{M}\,{}_m\mathcal{O}(K)$ such that $E_j(A_j) = S_j^{-1} L_j S_j$ for some $N \times N$ matrices S_j and L_j with $S_j \in \mathrm{GL}_N({}_{j-1}\mathcal{M}(K_{j-1}))$ and L_j strictly lower triangular with entries in ${}_{j-1}\mathcal{M}\,{}_j\mathcal{O}(K_j)$. Define the operators $T_j(\psi) = D_{\eta_j}\psi + \psi A_j$. Let $\{\phi_\alpha\}$ be a family of $1 \times N$ matrices with entries in ${}_m\mathcal{O}(K)$. Suppose that there exist indices $\alpha_1, \ldots, \alpha_q$ and $N \times 1$ matrices $Q_{i,j,k}$ with entries in ${}_m\mathcal{O}(K)$ for $1 \leq i \leq q$ and $1 \leq j, k \leq m$ such that $[D_{\eta_j} -$

$A_j, D_{\eta_k} - A_k] = \sum_i Q_{i,j,k}\phi_{\alpha_i}$. Let ${}_t\mathcal{N}(\psi) = \bigcap_{j_1,\ldots,j_t \geq 0} \ker E_t(T_t^{j_t}(\cdots T_2^{j_2}(T_1^{j_1}(\psi))\cdots)) \cap {}_t\mathcal{M}(K_t)^N$.

Theorem 3.5. *With the definitions and assumptions given in the immediately preceding paragraphs, it holds that*

$$\bigcap_\alpha {}_m\mathcal{N}(\phi_\alpha) = \left(\bigcap_\alpha \ker \phi_\alpha \cap \bigcap_{j=1}^m \ker(D_{\eta_j} - A_j)\right) \otimes_{\mathbb{C}} {}_m\mathcal{M}(K) \tag{3.14}$$

Proof. We will proceed using induction. Assume that

$$\bigcap_\alpha {}_t\mathcal{N}(E_t(\phi_\alpha)) = \left(\bigcap_\alpha \ker E_t(\phi_\alpha) \cap \bigcap_{j=1}^t \ker(D_{\eta_j} - E_t(A_j))\right) \otimes_{\mathbb{C}} {}_t\mathcal{M}(K_t), \tag{3.15}$$

with $1 \leq t < m$. (The base case $t = 1$ is an immediate consequence of Lemma 3.1 and equation (3.4).)

Let $\xi = \eta_{t+1}$. Let R denote the fundamental matrix for $(D_\xi - E_{t+1}(A_{t+1}))\vec{v}$ normalized at $\xi = \xi^*$. Using Lemma 3.1, Lemma 3.2, and Lemma 3.4, it holds that

$$\begin{aligned} &\cap_\alpha\, {}_{t+1}\mathcal{N}(E_{t+1}(\phi_\alpha)) \\ &= (\cap_\alpha\, {}_{t+1}\mathcal{N}(E_{t+1}(\phi_\alpha)) \cap \ker(D_\xi - E_{t+1}(A_{t+1}))) \otimes_{{}_t\mathcal{M}(K_t)} {}_{t+1}\mathcal{M}(K_{t+1}) \\ &= R\left(\cap_\alpha \cap_{j\geq 0}\, {}_t\mathcal{N}(E_t(T_{t+1}^j(\phi_\alpha)))\right) \otimes_{{}_t\mathcal{M}(K_t)} {}_{t+1}\mathcal{M}(K_{t+1}). \end{aligned} \tag{3.16}$$

By applying the inductive hypothesis to the larger family $\{E_t(T_{t+1}^j(\phi_\alpha))\}_{\alpha,j}$ and by Lemma 3.3 and equation (3.4), the above equals

$$\begin{aligned} &R\left(\cap_\alpha \cap_{j\geq 0} \ker(E_t(T_{t+1}^j(\phi_\alpha))) \cap \cap_{j=1}^t \ker(D_{\eta_j} - E_t(A_j)) \cap {}_t\mathcal{M}(K_t)^N\right) \\ &\qquad \otimes_{\mathbb{C}} {}_{t+1}\mathcal{M}(K_{t+1}) \\ &= \left(\cap_\alpha \ker E_{t+1}(\phi_\alpha) \cap \cap_{j=1}^{t+1} \ker(D_{\eta_j} - E_{t+1}(A_j)) \cap {}_{t+1}\mathcal{M}(K_{t+1})^N\right) \\ &\qquad \otimes_{\mathbb{C}} {}_{t+1}\mathcal{M}(K_{t+1}) \end{aligned} \tag{3.17}$$

By induction, (3.15) holds for $t = m$, thus the proof is complete. □

We carried out the previous results for a general family of $1 \times N$ matrices $\{\phi_\alpha\}$ because it actually facilitates the proof of Theorem 3.5. However Theorem 3.5 in the case where the family $\{\phi_\alpha\}$ simply consists of a single $1 \times N$ matrix ϕ is the main item of interest in our present application.

Notably, Theorem 3.5 implies that $\ker \phi \cap \bigcap_{j=1}^m \ker(D_{\eta_j} - A_j)$ is non-trivial if and only if ${}_m\mathcal{N}(\phi)$ is non-trivial. While ${}_m\mathcal{N}(\phi)$ is defined as the common null space of an infinite set of linear functionals dependent on ϕ, we will show via Theorem 3.6 that it can be calculated using only a finite subset of these linear functionals.

The following notation and definitions are adapted from [19]. Let $\mathcal{T} = \mathcal{T}^m$ denote the set of multi-indices, i.e. the set of m-tuples of non-negative integers. We consider lexicographical ordering on $\mathcal{T}$, using $\alpha \preceq \beta$ to denote that α equals or lexicographically precedes β. We also may define the partial ordering $\leq$, saying $(a_1, \ldots, a_m) \leq (b_1, \ldots, b_m)$ if and only if $a_j \leq b_j$ for all j.

We say that a subset A of $\mathcal{T}$ is *Young-like* if $\beta \leq \alpha$ for $\alpha \in A$ and $\beta \in \mathcal{T}$ implies that $\beta \in A$. A Young-like set corresponds to a m-dimensional partition with entries bounded by 1.

For $\alpha = (a_1, \ldots, a_m) \in \mathcal{T}$, we define $T^\alpha(\phi) = T_m^{a_m}(\cdots T_2^{a_2}(T_1^{a_1}(\phi))\cdots)$. For $A \subseteq \mathcal{T}$, we define $\mathcal{N}_A(\phi) = \bigcap_{\alpha \in A} \ker T^\alpha(\phi) \subseteq \mathcal{M}(K)^N$. (If A is Young-like, then $\mathcal{N}_A(\phi)$ also equals $\bigcap_{(a_1,\ldots,a_m)\in A} \ker T_1^{a_1}(T_2^{a_2}(\cdots T_m^{a_m}(\phi)\cdots))$, owing to Lemma 3.4.) For a finite subset $A \subseteq \mathcal{T}$, we use $M_A(\phi)$ (or $M_A^{T_1,T_2,\ldots,T_m}(\phi)$ should we wish to clearly identify $T_1, T_2, \ldots, T_m$) to denote the matrix with rows given by $\{T^\alpha(\phi)\}_{\alpha\in A}$, listed in lexicographical order. Thus $\mathcal{N}_A(\phi)$ is the null space of $M_A(\phi)$. If A has cardinality N, then let $W_A(\phi) = \det M_A(\phi)$ (and likewise let $W_A^{T_1,T_2,\ldots,T_m}(\phi) = \det M_A^{T_1,T_2,\ldots,T_m}(\phi)$).

The following is an generalization of Lemma 3.2 of [19], with the proof following in the same spirit.

Theorem 3.6. *For a given $1 \times N$ matrix ϕ, there exists a Young-like set $Y \subset \mathcal{T}$ with cardinality at most N, such that $\mathcal{N}_Y(\phi) = \mathcal{N}_\mathcal{T}(\phi)$.*

Proof. Let $S_\alpha = \{\beta \in \mathcal{T} \mid \beta \prec \alpha\}$ and $Y = \{\alpha \in \mathcal{T} \mid \mathcal{N}_{S_\alpha} \neq \mathcal{N}_{S_\alpha \cup \{\alpha\}}\}$. Since $\mathcal{N}_{S_\alpha\cup\{\alpha\}}$ is a proper $\mathcal{M}(K)$ vector subspace of $\mathcal{N}_{S_\alpha}$ for $\alpha \in Y$ and since $\mathcal{N}_{S_\beta} \subseteq \mathcal{N}_{S_\alpha}$ when $\alpha \prec \beta$, we see that Y must have cardinality at most N, as otherwise there would exist an $\alpha \in \mathcal{T}$ such that $\mathcal{N}_{S_\alpha}$ had negative dimension.

We claim that $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\alpha\cup\{\alpha\}}$ for all $\alpha \in \mathcal{T}$. This may be shown by induction. Let $\beta \in \mathcal{T}$ and assume that the claim holds for all $\alpha \prec \beta$, which implies that $\mathcal{N}_Y \subseteq \cap_{\alpha\prec\beta}\mathcal{N}_{S_\alpha\cup\{\alpha\}} = \mathcal{N}_{S_\beta}$. Since $\mathcal{N}_Y \subseteq \ker T^\beta(\phi)$ if $\beta \in Y$ and since $\mathcal{N}_{S_\beta} = \mathcal{N}_{S_\beta\cup\{\beta\}}$ if $\beta \notin Y$, it follows that $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\beta\cup\{\beta\}}$. So the claim holds, and so it follows that $\mathcal{N}_Y = \mathcal{N}_\mathcal{T}$.

Now let $\alpha \in \mathcal{T}\backslash Y$, $\beta \in \mathcal{T}$, and $\alpha \prec \beta$. As $\alpha \notin Y$, it follows that there exist $c_\gamma \in \mathcal{M}(K)$ for $\gamma \in S_\alpha$, only finitely many being non-zero, such that

$$T^\alpha(\phi) = \sum_{\gamma\in S_\alpha} c_\gamma T^\gamma(\phi). \tag{3.18}$$

In light of Lemma 3.4 and relations in the form of (3.2), we may apply $T^{\beta-\alpha}$ to this equation to see that $T^\beta(\phi)$ is a $\mathcal{M}(K)$ linear combination of the elements of $\{T^\gamma(\phi)\}_{\gamma\in S_\beta}$. Thus Y is Young-like.

□

Note that a Young-like set with cardinality less than N is contained in some Young-like set of cardinality exactly N. So Theorem 3.6 yields the following two corollaries.

Corollary 3.7. *For a* $1\times N$ *matrix* ϕ, $\mathcal{N}_{\mathcal{T}}(\phi)$ *is non-trivial if and only if* $W_Y(\phi) = 0$ *for every Young-like set* Y *of cardinality* N.

Corollary 3.8. *Let* $\hat{Y}$ *denote the union of all Young-like sets of cardinality* N. $\hat{Y}$ *is a Young-like set of finite cardinality at least* N *and* $\mathcal{N}_{\hat{Y}}(\phi) = \mathcal{N}_{\mathcal{T}}(\phi)$ *for all* $1\times N$ *matrices* ϕ.

Coupling either of these corollaries with Theorem 3.5 would produce criteria for characterizing those ϕ for which $\ker\phi \cap \bigcap_{j=1}^m \ker(D_{\eta_j} - A_j)$ is non-trivial. For instance, we may form the following from Theorem 3.5 and Corollary 3.7

Corollary 3.9. *With the assumptions of Theorem 3.5 and with* ϕ *being a* $1 \times N$ *matrix with entries in* ${}_m\mathcal{O}(K)$, *there exists a non-trivial element in* $\ker\phi \cap \bigcap_{j=1}^m \ker(D_{\eta_j} - A_j)$ *if and only if* $W_Y(\phi) = 0$ *for every Young-like set* Y *of cardinality* N.

4. Meromorphic Whitney Multifunction Solutions to the Shockwave Equation

Consider the equation

$$f f_\xi = f_\eta, \tag{4.1}$$

for a complex function f on a domain in $\mathbb{C}^2$ with coordinates (ξ, η). For simplicity, we will refer to (4.1) as the *shockwave equation.* (This equation can also be described the complexified, sign-flipped Burgers equation without viscosity, with ξ corresponding to the space variable and η corresponding to the time variable.)

A *complex multifunction on* X*, in the sense of Whitney*, or a *complex Whitney multifunction on* X, is a map from X to $\mathbb{C}^m_{sym}$ for some m, where $\mathbb{C}^m_{sym}$ denotes the mth symmetric set power of $\mathbb{C}$ [22]. (One could also view elements of $\mathbb{C}^m_{sym}$ as positive divisors on $\mathbb{C}$ with degree m.) Complex Whitney multifunctions have a natural correspondence with the ζ-root systems of the polynomials

$$\zeta^N - e_1(x)\zeta^{N-1} + \cdots + (-1)^N e_N(x), \tag{4.2}$$

where $e_j(x)$ is a function on X giving the jth elementary symmetric function of the outputs of the Whitney multifunction. A *holomorphic Whitney multifunction* on X is a complex Whitney multifunction whose elementary symmetric functions are holomorphic over X. So there is natural correspondence between holomorphic Whitney multifunctions and Weierstrass polynomials.

Terminological Note: The term *multifunction* is often used to denote any set-valued function, which is different from a Whitney multifunction. Likewise a *holomorphic multifunction*, as it is customarily defined, differs from a holomorphic Whitney multifunction. Somewhat related to a holomorphic Whitney multifunction is an *algebroid multifunction* on X, which denotes a set-valued function of the form

$$F(x) = \{\zeta \in \mathbb{C} \mid \zeta^n + a_1(x)\zeta^{n-1} + \cdots + a_n(x) = 0\}, \tag{4.3}$$

where $a_1, a_2, \ldots, a_n$ are holomorphic functions on X. However even this differs from a holomorphic Whitney multifunction, for an algebroid multifunction treats the roots as a set without regarding multiplicity. For example, let $f(x)$ be a holomorphic function, then $(\zeta - f(x))$ and $(\zeta - f(x))^2$ produce two distinct holomorphic Whitney multifunctions, yet they correspond to the same algebroid multifunction.

We say that $e_1, e_2, \ldots, e_N$ satisfy the *multi-shockwave* system of equations if

$$(e_1)_\xi e_k - (e_{k+1})_\xi = (e_k)_\eta, \text{ for } 1 \le k \le N, \tag{4.4}$$

where e_{N+1} is regarded as 0. This definition stems from the following.

Lemma 4.1. *Let $f_1, f_2, \ldots, f_N$ be continuous functions defined on a domain Ω. Let e_k be the kth elementary symmetric function of $f_1, f_2, \ldots, f_N$. The following are equivalent.*

1. *The functions $f_1, f_2, \ldots, f_N$ are holomorphic and each satisfy the shockwave equation (4.1) on Ω.*
2. *The functions $e_1, e_2, \ldots, e_N$ are holomorphic and satisfy the multi-shockwave system of equations (4.4) on Ω.*

Proof. Assume 1. The functions $\{e_k\}$ are clearly holomorphic and the multi-shockwave equations follow since

(4.5)

$$\begin{aligned}(e_1)_\xi e_k - (e_{k+1})_\xi &= \sum_{i_1 < \cdots < i_k} \sum_{\ell=1}^{N} f_{i_1} \cdots f_{i_k} (f_\ell)_\xi - \sum_{i_1 < \cdots < i_k} \sum_{\ell \notin \{i_1, \ldots, i_k\}} f_{i_1} \cdots f_{i_k} (f_\ell)_\xi \\ &= \sum_{i_1 < \cdots < i_k} \sum_{\ell \in \{i_1, \ldots, i_k\}} (f_\ell)_\eta f_{i_1} \cdots \hat{f}_\ell \cdots f_{i_k} = (e_k)_\eta.\end{aligned}$$

Now assume 2. The functions $f_1, f_2, \ldots, f_N$ are roots of the polynomial

$$\zeta^N - e_1\zeta^{N-1} + \cdots + (-1)^N e_N = \prod_{j=1}^{N} (\zeta - f_j) \tag{4.6}$$

and are holomorphic, due to the holomorphicity of $e_1, e_2, \ldots, e_N$, [22] (pg. 27, Theorem 9D).

Let (ξ^*, η^*) be an arbitrary point in Ω. Let $\tilde{f}_j = f_j|_{\eta=\eta^*}$ and $\tilde{e}_k = e_k|_{\eta=\eta^*}$, which are holomorphic functions of ξ on $\tilde{\Omega} = \Omega \cap \{\eta = \eta^*\}$. By the Cauchy-Kovalevski Theorem [8](pg. 16), there exist holomorphic functions $F_1, F_2, \dots, F_N$ satisfying the shockwave equation on some neighborhood Ω' of (ξ^*, η^*) in Ω with the initial condition $F_j = \tilde{f}_j$ on $\tilde{\Omega} \cap \Omega'$. Let E_k be the kth elementary symmetric function of $F_1, F_2, \dots, F_N$. It follows by (4.5) that $E_1, E_2, \dots, E_N$ satisfy the multi-shockwave equations. Since $E_k = \tilde{e}_k$ on $\tilde{\Omega} \cap \Omega'$, it holds, due to uniqueness in the Cauchy-Kovalevski theorem, that E_j equals e_j on Ω' for every j. Consequentially, each f_j equals F_j and so satisfies the shockwave equation. □

Holomorphic Whitney multifunctions are locally unramified away from the set of branch points, which is an analytic set with complex codimension one. So it is altogether fitting to say that holomorphic solutions to the multi-shockwave equations represent holomorphic Whitney multifunction solutions to the shockwave equation. (Note: A like form of the multi-shockwave equations, involving $(-1)^k e_k$ instead, has also been developed in [12](Lemma 16), though specifically in the case of a non-trivial discriminant, i.e. the case of an algebroid multifunction solution.)

For the notion of meromorphic Whitney multifunction solutions to the shockwave equation, we consider the elementary symmetric functions $(e_1, e_2, \dots, e_N)$ in the homogeneous (or projective) form $[P_0 : P_1 : \cdots : P_N]$, where e_k is corresponds to P_k/P_0. We say that $P_0, P_1, \dots, P_N$ satisfy the *general homogenized multi-shockwave* system of equations or the *general h.s.w.* system of equations, for the sake of conciseness, if

$$
\begin{aligned}
(4.7) \quad & P_0[(P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi + (P_k)_\eta P_0] \\
& \qquad = P_k[(P_1)_\xi P_0 - P_1(P_0)_\xi + (P_0)_\eta P_0] \text{ for } 1 \le k \le N,
\end{aligned}
$$

where P_{N+1} is regarded as 0. The justification for this definition is the following.

Lemma 4.2. *Let $e_1, e_2, \dots, e_N$ and P_0 be functions defined on a domain Ω. Assume that P_0 is not identically zero and let $P_k = e_k P_0$. The following are equivalent.*

(1) *The functions $e_1, e_2, \dots, e_N$ and P_0 are holomorphic and $e_1, e_2, \dots, e_N$ satisfy the multi-shockwave system of equations (4.4) on Ω.*
(2) *The functions $P_0, P_1, \dots, P_N$ are holomorphic, P_0 divides each function P_j within $\mathcal{O}(\Omega)$, and $P_0, P_1, \dots, P_N$ satisfy the general h.s.w. system of equations (4.7) on Ω.*

Proof. It is clear that $e_1, e_2, \dots e_N$ and P_0 are holomorphic if and only if $P_0, P_1, \dots, P_N$ are holomorphic and each P_j, for $1 \le j \le N$, is divisible by P_0 in $\mathcal{O}(\Omega)$.

Completion of this proof only requires the calculation

$$
\begin{aligned}
&\left[\left(\frac{P_1}{P_0}\right)_\xi \frac{P_k}{P_0} - \left(\frac{P_{k+1}}{P_0}\right)_\xi - \left(\frac{P_k}{P_0}\right)_\eta\right] P_0^3 \\
&= ((P_1)_\xi P_0 - P_1(P_0)_\xi)P_k - ((P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi)P_0 - ((P_k)_\eta P_0 - P_k(P_0)_\eta)P_0 \\
&= P_k[(P_1)_\xi P_0 - P_1(P_0)_\xi + (P_0)_\eta P_0] - P_0[(P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi + (P_k)_\eta P_0].
\end{aligned}
$$

□

In analogy to the homogeneous coordinates for projective space, let $[P_0 : P_1 : \cdots : P_N]$ denote the equivalence class in $\mathcal{O}(\Omega)^{N+1}\backslash\{0\}$ under the equivalence relation $\sim$ where $(P_0, P_1, \ldots, P_N) \sim (Q_0, Q_1, \ldots, Q_N)$ if there exists a meromorphic function λ not equivalently zero such that $Q_j = \lambda P_j$ for $0 \leq j \leq N$. (One could also consider the corresponding equivalence class in $\mathcal{M}(\Omega)^{N+1}\backslash\{0\}$.) As one may verify, it is well-defined to say that $[P_0 : P_1 : \cdots : P_N]$ satisfies (or does not satisfy) the general h.s.w. equations, according to whether any or every representative of the equivalence class does.

Assumptions on K: For the duration of this section we assume that K is a compact set containing a point (ξ^*, η^*) such that

(a) $\mathcal{O}(K)$ is a unique factorization domain,
(b) non-empty slices of the form $K \cap \{\eta = \eta_0\}$, for $\eta_0 \in \mathbb{C}$, are connected and contain the point (ξ^*, η_0), and
(c) $\tilde{K} = \{\eta \in \mathbb{C} \mid (\xi^*, \eta) \in K\}$ is Cauchy-viable with respect to $\eta = \eta^*$ (i.e. it possesses a neighborhood basis of simply connected domains.)

Simple yet noteworthy examples of sets K with these properties include the point (ξ^*, η^*) and any closed polydisk containing (ξ^*, η^*).

Our previous definitions and results, which were given for $\mathcal{O}(\Omega)$, also hold with $\mathcal{O}(K)$ instead. Our main reason for focusing on $\mathcal{O}(K)$ rather than $\mathcal{O}(\Omega)$ is that the latter cannot be a unique factorization domain.

We call $(P_0, P_1, \ldots, P_N)$ a lowest terms representative of $[P_0 : P_1 : \cdots : P_N]$ if $P_0, P_1, \ldots, P_N$ have no common irreducible factors in $\mathcal{O}(K)$. By our assumptions on K, a lowest terms representative exists and is unique, up to multiplication by a unit in $\mathcal{O}(K)$. The following reveals a useful subclass of lowest terms representatives for solutions of (4.7).

Lemma 4.3. *Let $Q_0, Q_1, \ldots, Q_N \in \mathcal{O}(K)$, with Q_0 not identically zero and K satisfying the assumed properties of this section. If $[Q_0 : Q_1 : \cdots : Q_N]$ satisfies the general h.s.w. equations (4.7), then there exist $P_0, P_1, \ldots, P_N \in \mathcal{O}(K)$, with P_0 not identically zero, such that $(P_0)_\xi = 0$ and $(P_0, P_1, \cdots, P_N)$ is a lowest terms representative of $[Q_0 : Q_1 : \cdots : Q_N]$.*

Proof. Assume for sake of contradiction that there is no lowest terms representation satisfying the conclusion of the lemma. Let $(P_0, P_1, \cdots, P_N)$ be a lowest terms representation of $[Q_0 : Q_1 : \cdots : Q_N]$. Then there exists an irreducible r in $\mathcal{O}(K)$ that divides P_0 such that $(ur)_\xi \not\equiv 0$ for every unit u in $\mathcal{O}(K)$. Let n be the largest positive integer such that $r^n | P_0$, and let k be the smallest positive integer such that $r \not| P_k$. Let $\alpha = P_0(P_1)_\xi - (P_0)_\xi P_1 + P_0(P_0)_\eta$ and note that

$$P_0[(P_{k+1})_\xi P_0 - P_{k+1}(P_0)_\xi + (P_k)_\eta P_0] = P_k \alpha. \tag{4.8}$$

Since r^{2n-1} divides the left hand side, it divides α. Next note that

$$P_{k-1}\alpha = (P_k)_\xi P_0^2 - P_k(P_0)_\xi P_0 + (P_{k-1})_\eta P_0^2, \tag{4.9}$$

which holds due to the general h.s.w. equations if $k > 1$ or holds tautologically if $k = 1$. The left-hand side is divisible by r^{2n}, as are the first and third terms on the right-hand side. So $r^{2n} | P_k(P_0)_\xi P_0$, and thus $r^n | (P_0)_\xi$. Applying the product rule to a factorization of P_0 shows that $r | r_\xi$. Therefore $r_\xi = cr$ for some c in $\mathcal{O}(K)$.

Let (ξ_0, η_0) be a point in K where r vanishes. (If no such point exists, then r is an unit.) On some neighborhood of (ξ_0, η_0), it holds that $r(\xi, \eta) = r(\xi_0, \eta) \exp(\int_{\xi_0}^{\xi} c(\xi', \eta)\, d\xi')$. Thus r vanishes along $\eta = \eta_0$ near (ξ_0, η_0). Since $K \cap \{\eta = \eta_0\}$ is connected, it follows that r is zero along $K \cap \{\eta = \eta_0\}$. As r is irreducible, this implies that r is the product of $(\eta - \eta_0)$ and a unit in $\mathcal{O}(K)$, achieving the desired contradiction.

□

Remark: As one consequence, Lemma 4.3 shows that the pole set of a meromorphic Whitney multifunction shockwave solution on K lies in a finite union of lines of the form $\{\eta = \eta_j\}$.

We say that $P_0, P_1, \ldots, P_N$ satisfy the *refined homogenized multi-shockwave* system of equations, or the *refined h.s.w.* system of equations, if

$$P_0[(P_{k+1})_\xi + (P_k)_\eta] = P_k[(P_1)_\xi + (P_0)_\eta], \text{ for } 1 \le k \le N, \text{ and } \quad (P_0)_\xi = 0, \tag{4.10}$$

where P_{N+1} is regarded as 0. We say that $P_0, P_1, \ldots, P_N$ satisfy the *special homogenized multi-shockwave* system of equations, or the *special h.s.w.* system of equations, if there exists a holomorphic function μ such that

$$(P_{k+1})_\xi + (P_k)_\eta = \mu P_k, \text{ for } 0 \le k \le N, \text{ and } \qquad (P_0)_\xi = 0, \tag{4.11}$$

where P_{N+1} is regarded as 0.

When $(P_0)_\xi = 0$, the general h.s.w. equations reduce to the refined h.s.w. equations. So Lemma 4.3 shows that every solution to the general h.s.w. equations (4.7) in $\mathcal{O}(K)$ has a lowest terms representative that satisfies the refined h.s.w. equations (4.10). This particular representative can be described in a number of other equivalent ways.

Lemma 4.4. *Let $P_0, P_1, \ldots, P_N \in \mathcal{O}(K)$, with P_0 not identically zero and K satisfying the assumed properties of this section. The following are equivalent.*

(1) *$P_0, P_1, \ldots, P_N$ satisfy the refined h.s.w. equations, and $P_0, P_1, \ldots, P_N$ have no common irreducible factors in $\mathcal{O}(K)$.*
(2) *$P_0, P_1, \ldots, P_N$ satisfy the refined h.s.w. equations and P_0 divides $((P_1)_\xi + (P_0)_\eta)$ in $\mathcal{O}(K)$.*
(3) *$P_0, P_1, \ldots, P_N$ satisfy the special h.s.w. equations.*

Proof. First we establish the equivalence of 1 and 2. Assume 1 and let r be any irreducible factor that divides P_0 with a positive multiplicity m. There exists a $k \geq 1$ such that $r \not| P_k$. Thus, by the refined h.s.w. equations (4.10), r divides $((P_1)_\xi + (P_0)_\eta)$, also with multiplicity m. It follows that P_0 divides $(P_1)_\xi + (P_0)_\eta$, implying 2.

Assume 2. Let $(R_0, R_1, \ldots, R_N)$ be a lowest terms representation of $[P_0 : P_1 : \cdots : P_N]$ satisfying 1, which exists by Lemma 4.3. So there exists a non-zero $\lambda \in \mathcal{O}(K)$ such that $P_i = \lambda R_i$, for all i. Since $(R_0)_\xi = 0$ and $(P_0)_\xi = 0$, it follows that $\lambda_\xi = 0$. By the previous paragraph, the functions $R_0, R_1, \ldots, R_N$ also satisfy 2. By using that both $R_0 | ((R_1)_\xi + (R_0)_\eta)$ and $P_0 | ((P_1)_\xi + (P_0)_\eta)$ we obtain that $\lambda | \lambda_\eta$. It follows that λ is non-vanishing and therefore a unit in $\mathcal{O}(K)$. Thus 1 holds.

To conclude we establish the equivalence of 2 and 3. Assuming 2, it follows that $\mu = \frac{(P_1)_\xi + (P_0)_\eta}{P_0}$ is an holomorphic function. Then (4.11) holds for $k = 0$ tautologically, and it holds for other k by dividing each of the refined h.s.w. equations (4.10) by P_0. So 3 follows.

Assuming 3, then (4.11) for $k = 0$ gives that $(P_1)_\xi + (P_0)_\eta = \mu P_0$. Therefore $P_0 | ((P_1)_\xi + (P_0)_\eta)$ and the refined h.s.w. equations (4.10) follow by multiplying the special h.s.w. equations (4.11) by P_0.

□

Let $P_0, P_1, \ldots, P_N \in \mathcal{O}(K)$, with P_0 not identically vanishing, such that $[P_0 : P_1 : \cdots : P_N]$ satisfies the general h.s.w. equations. We call $(P_0, P_1, \ldots, P_N)$ a *refined representative* of $[P_0 : P_1 : \cdots : P_N]$, if $(P_0)_\xi = 0$. We call $(P_0, P_1, \ldots, P_N)$ a *special representative* of $[P_0 : P_1 : \cdots : P_N]$ if it satisfies any one of the equivalent conditions in Lemma 4.4.

If $(P_0, P_1, \ldots, P_N)$ is a special representative of $[P_0 : P_1 : \cdots : P_N]$, then all other special representatives are of the form $(\lambda P_0 : \lambda P_1 : \cdots : \lambda P_N)$ where $\lambda_\xi = 0$ and λ is a unit in $\mathcal{O}(K)$. (One may see this in the proof of Lemma 4.4.)

Suppose that $P_0, P_1, \ldots, P_N$ satisfy the special h.s.w. equations for a particular μ. Let λ be a unit in $\mathcal{O}(K)$ such that $\lambda_\xi = 0$, and let $\hat{P}_k = \lambda P_k$ for $0 \leq k \leq N$. Then $\hat{P}_0, \hat{P}_1, \ldots, \hat{P}_N$ satisfy the special h.s.w. equations with $\hat{\mu} = \mu + \frac{\lambda_\eta}{\lambda}$ in place of μ. Observe that $\hat{\mu}_\xi = \mu_\xi$. So μ_ξ remains unchanged, whereas $\mu|_{\xi=\xi^*}$ can be modified. If we let $\lambda(\xi,\eta) = \exp(-\int_{\eta^*}^{\eta} \mu(\xi^*,\eta')\, d\eta')$, which is well-defined due to our assumptions on K, then $\hat{\mu}|_{\xi=\xi^*} = 0$.

If $P_0, P_1, \ldots, P_N$ satisfy the special h.s.w. equations with a function μ such that $\mu|_{\xi=\xi^*} = 0$, then we say that $P_0, P_1, \ldots, P_N$ satisfy the *canonical h.s.w.* equations and that $(P_0, P_1, \ldots, P_N)$ is a *canonical representative* of $[P_0 : P_1 : \cdots : P_N]$. Two canonical representatives differ only by multiplication by a complex number. This discussion yields the following.

Proposition 4.5. *With K satisfying the assumptions of this section, any solution $P_0, P_1, \ldots, P_N$ to the general h.s.w. equations in $\mathcal{O}(K)$, with P_0 not identically zero, has a canonical representative.*

5. An Extended Shockwave Characterization of Boundaries of Holomorphic 1-Chains within $\mathbb{CP}^2$

The focal point of this section is Theorem 5.1, which enhances the Dolbeault and Henkin characterization for the case of $\mathbb{CP}^2$ by employing meromorphic multi-shockwaves in place of unramified holomorphic shockwaves.

Let $(w_0 : w_1 : w_2)$ denote homogeneous coordinates for $\mathbb{CP}^2$. We identify $\mathbb{C}^2$ with the standard affine portion of $\mathbb{CP}^2$, given by $w_0 \neq 0$, on which we may use the affine coordinates $(z_1, z_2) = (w_1/w_0, w_2/w_0)$. We use the Fubini-Study metric on $\mathbb{CP}^2$ for the purposes as defining k-dimensional Hausdorff measure on $\mathbb{CP}^2$ and the mass seminorms for currents on $\mathbb{CP}^2$.

Let γ be a closed, rectifiable 1-current whose support is contained in $\mathbb{C}^2$ and satisfies condition A_1. The definition of rectifiable currents can be found in a number of sources, such as the treatise by Federer [6] or the article by Harvey [10] which is well-geared to the context of this paper. For the definition of condition A_1, one may refer to the work of Dinh [3].

Let V be a holomorphic 1-chain in $\mathbb{CP}^2 \setminus \operatorname{spt}\gamma$, and suppose that V has a trivial extension to a current in $\mathbb{CP}^2$. Viewing V as a current in $\mathbb{CP}^2$, we say that γ *bounds a holomorphic 1-chain V within* $\mathbb{CP}^2$ or that γ is the *boundary of V within* $\mathbb{CP}^2$ if $dV = \gamma$.

Notes: (1) In order for V to have a trivial extension, it is sufficient that V have finite mass [13]. (2) If γ bounds V within $\mathbb{CP}^2$ then, unless γ is zero, V is not a genuine holomorphic 1-chain in $\mathbb{CP}^2$. References to V as a holomorphic 1-chain are correct when they are interpreted in $\mathbb{CP}^2 \setminus \operatorname{spt}\gamma$.

Let $g_{\xi,\eta} = z_2 - \xi - \eta z_1$ and $\tilde{g}_{\xi,\eta} = w_2 - \xi w_0 - \eta w_1$. The pair (ξ,η) can be viewed as coordinates for an affine portion of $(\mathbb{CP}^2)'$, via correspondence to the line $g_{\xi,\eta} = 0$. Define the projection $\pi_\eta : \mathbb{C}^2 \to \mathbb{C}$ by $(z_1, z_2) \mapsto z_2 - \eta z_1$, and let

$\mathfrak{U}_\gamma = \{(\xi,\eta) \in \mathbb{C}^2 \mid \xi \notin \pi_\eta(\gamma)\}$. For $(\xi,\eta) \in \mathfrak{U}_\gamma$, we define

$$G_\gamma(\xi,\eta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma z_1 \frac{dg_{\xi,\eta}}{g_{\xi,\eta}} = \frac{1}{2\pi \mathrm{i}} \int_\gamma z_1 \frac{d(z_2 - \eta z_1)}{z_2 - \xi - \eta z_1}. \tag{5.1}$$

Theorem 5.1. *For a closed, rectifiable 1-current γ whose support is contained in $\mathbb{C}^2 \subset \mathbb{CP}^2$ and satisfies condition A_1, the following are equivalent:*

(i) *γ bounds a holomorphic 1-chain, with finite mass, within $\mathbb{CP}^2$.*

(ii) *$\exists$ a point (ξ^*,η^*) with a neighborhood Ω for which there exist non-negative integers N^+ and N^- and holomorphic functions f_j^+ for $1 \le j \le N^+$ and f_j^- for $1 \le j \le N^-$ on Ω, each satisfying the shockwave equation (4.1), such that*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2} \left(\sum_{j=1}^{N^+} f_j^+(\xi,\eta) - \sum_{j=1}^{N^-} f_j^-(\xi,\eta) \right). \tag{5.2}$$

(iii) *$\exists$ a point (ξ^*,η^*) with a neighborhood Ω for which there exist non-negative integers N^+ and N^- and holomorphic functions e_k^+ for $1 \le k \le N^+$ and e_k^- for $1 \le k \le N^-$ on Ω, with both lists of functions satisfying the multi-shockwave system of equations (4.4), such that*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2} \left(e_1^+(\xi,\eta) - e_1^-(\xi,\eta) \right). \tag{5.3}$$

(iii') *$\exists$ an open set $U \subseteq \mathbb{C}$ such that any point (ξ^*,η^*) with any neighborhood domain $\Omega \subseteq \mathfrak{U}_\gamma \cap (\mathbb{C} \times U)$ satisfies the criterion given in (iii).*

(iv) *$\exists$ a point (ξ^*,η^*) with a neighborhood Ω for which there exist non-negative integers N^+ and N^- and holomorphic functions $P_k^+(\xi,\eta)$ for $0 \le k \le N^+$ and $P_k^-(\xi,\eta)$ for $0 \le k \le N^-$ on Ω, with neither P_0^+ nor P_0^- being identically zero and both lists of functions satisfying the refined h.s.w. system of equations (4.10), such that*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial \xi^2} \left(\frac{P_1^+(\xi,\eta)}{P_0^+(\xi,\eta)} - \frac{P_1^-(\xi,\eta)}{P_0^-(\xi,\eta)} \right). \tag{5.4}$$

(iv') *Any point (ξ^*,η^*) with any neighborhood domain $\Omega \subseteq \mathfrak{U}_\gamma$ satisfies the criterion given in (iv).*

Dolbeault and Henkin, in addressing boundaries of holomorphic 1-chains within $\mathbb{CP}^n$, originally established the equivalence of conditions (i) and (ii) in the case that γ is a closed, oriented, $\mathcal{C}^2$ 1-chain in $\mathbb{C}^2$, and without taking the second partial derivatives on either side of the decomposition (5.2) [4]. In their subsequent work, in which they were considering boundaries of holomorphic p-chains within $\mathbb{CP}^n$, they introduce decompositions modulo ξ-affine functions [5], which produces a more naturally equivalent statement. Dinh relaxed the regularity required for γ to the assumptions presented here [3].

We note a corollary and some applications of Theorem 5.1 before proceeding to its proof.

Corollary 5.2. *Let γ be a closed, rectifiable 1-current whose support is contained in $\mathbb{C}^2 \subset \mathbb{CP}^2$ and satisfies condition A_1, and let $(\xi^*, \eta^*) \in \mathfrak{U}_\gamma$. The following are equivalent:*

(1) *γ bounds a holomorphic 1-chain, with finite mass, within $\mathbb{CP}^2$*

(2) *There exist germs of holomorphic functions $P_0^+, P_1^+, \ldots, P_{N^+}^+$ and $P_0^-, P_1^-, \ldots, P_{N^-}^-$ at (ξ^*, η^*), for some non-negative integers N^+ and N^-, with P_0^+ and P_0^- not identically zero and both lists satisfying the refined h.s.w. system of equations (4.10), such that*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi, \eta) = \frac{\partial^2}{\partial \xi^2} \left(\frac{P_1^+(\xi, \eta)}{P_0^+(\xi, \eta)} - \frac{P_1^-(\xi, \eta)}{P_0^-(\xi, \eta)} \right) \tag{5.5}$$

near (ξ^*, η^*).

Remark: Replacing the refined h.s.w. equations in 2 of Corollary 5.2 with either the general h.s.w., special h.s.w., or canonical h.s.w. equations would produce additional equivalent statements.

We return our attention to Theorem 5.1. If (i) holds then the points (ξ^*, η^*) at which statement (ii) holds is the complement of an analytic set in $\mathfrak{U}_\gamma$ dependent on the family of holomorphic 1-chains bounded by γ. (cf. Lemma 5.3.) It is not immediately clear whether this analytic set could be directly discerned from γ without knowing the family of holomorphic 1-chains bounded by γ. Of course, if one knows the whole family of holomorphic 1-chains bounded by γ, then it is moot to check (ii) to determine whether (i) holds. If we exclude such a priori knowledge, then verifying (ii) would seem to require some serendipity (albeit generic serendipity) in selecting (ξ^*, η^*) or, inversely, contradicting (ii) would require the consideration of a suitably broad range of (ξ^*, η^*). In contrast, condition (iv), being paired with (iv'), is free of such hidden obstructions, which permits one to arbitrarily fix (ξ^*, η^*) in $\mathfrak{U}_\gamma$. (Note that $\mathfrak{U}_\gamma$ depends on γ in a transparent fashion.)

One can characterize whether γ bounds a holomorphic 1-chain having prescribed behavior near the line $z_2 = \xi^* + \eta^* z_1$ according to whether G_γ has a decomposition at (ξ^*, η^*) bearing a corresponding set of constraints. This method follows from the Darboux Lemma, i.e. Lemma 5.3, its various extensions, i.e. Lemma 5.4 and Lemma 5.5, and their converse. (For example, conditions on the degree of positive and negative intersections with the line translate into conditions on the degree of the positive and negative portions of the decomposition.) One may do this with decompositions in the form of either (ii), (iii), or (iv). But when we fix our attention on a specific line $z_2 = \xi^* + \eta^* z_1$, then decompositions of the form in (ii) make some impositions on the choice of prescribed behavior. (Specifically, this forces the requirement that intersections of the holomorphic 1-chain with the line $z_2 = \xi^* + \eta^* z_1$ are locally component-wise transverse and that they do not occur at infinity.) By using (iv) we avoid these impositions, thus it grants us a general choice in prescribing behavior near the line $z_2 = \xi^* + \eta^* z_1$.

The remainder of this section is devoted to the proof of Theorem 5.1. For a holomorphic 1-chain V bounded by γ, let $\mathcal{T}_V$ be the set of all $(\xi, \eta) \in \mathcal{U}_\gamma$ such that some local component of V is not transverse to the line $w_2 = \xi w_0 + \eta w_1$. Also let $\mathcal{I}_V$ to be the set of all $(\xi, \eta) \in \mathcal{U}_\gamma$ such that the line $w_2 = \xi w_0 + \eta w_1$ intersects a component of V at the line at infinity $w_0 = 0$.

Lemma 5.3. *Let V be a positive holomorphic 1-chain with finite mass bounded by γ within $\mathbb{CP}^2$ and suppose that V has no components in the line at infinity $w_0 = 0$. For any simply-connected domain $\Omega \subseteq \mathcal{U}_\gamma \backslash (\mathcal{T}_V \cup \mathcal{I}_V)$, there exists a nonnegative integer N and holomorphic functions $f_1, f_2, \ldots, f_N$ satisfying the shockwave equation (4.1) on Ω, such that*

$$\frac{\partial^2}{\partial \xi^2} G_\gamma(\xi, \eta) = \frac{\partial^2}{\partial \xi^2} \sum_{j=1}^{N} f_j(\xi, \eta). \tag{5.6}$$

This lemma can also be called the Darboux Lemma. The following proof is similar to that given by Dolbeault and Henkin [4](Lemme 2.3).

Proof. Let N denote the total degree of intersection between V and the line $w_2 = \xi w_0 + \eta w_1$, which is constant for (ξ, η) in Ω. As Ω is simply connected and disjoint from $\mathcal{T}_V \cup \mathcal{I}_V$, we may define holomorphic maps $p_j(\xi, \eta)$ from Ω to $\mathbb{C}^2$, for $1 \leq j \leq N$, such that $p_1(\xi, \eta), p_2(\xi, \eta), \ldots, p_N(\xi, \eta)$ are the points of intersection, counting multiplicity, between V and $w_2 = \xi w_0 + \eta w_1$. Let $f_j(\xi, \eta) = z_1|_{p_j(\xi, \eta)}$.

Let $f = f_j$ and $p = p_j$ for some j. Let $(\xi_0, \eta_0) \in \Omega$, and define $h = f(\xi_0, \eta_0)$. The point $p(\xi_0, \eta_0) = (h, \xi_0 + \eta_0 h)$ is in the intersection of V and $\{z_2 = (\xi_0 - \tau h) + (\eta_0 + \tau) z_1\}$ for all τ such that $(\xi_0 - \tau h, \eta_0 + \tau) \in \Omega$. Furthermore,

$$f(\xi_0 - \tau f(\xi_0, \eta_0), \eta_0 + \tau) = f(\xi_0, \eta_0), \tag{5.7}$$

for τ near 0. Differentiation with respect to τ and evaluation at $\tau = 0$ of the above yields that

$$f_\xi(\xi_0, \eta_0)(-f(\xi_0, \eta_0)) + f_\eta(\xi_0, \eta_0) = 0. \tag{5.8}$$

As (ξ_0, η_0) was a general point in Ω, we see that $f f_\xi = f_\eta$ on Ω.

It only remains to establish (5.6). Note that

$$G_\gamma(\xi, \eta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{w_1}{w_0} \frac{d(\tilde{g}_{\xi,\eta}/w_0)}{\tilde{g}_{\xi,\eta}/w_0} = \frac{1}{2\pi \mathrm{i}} \int_\gamma \left(\frac{w_1 d\tilde{g}_{\xi,\eta}}{w_0 \tilde{g}_{\xi,\eta}} - \frac{w_1 dw_0}{w_0^2} \right). \tag{5.9}$$

By residue calculations,

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{w_1 d\tilde{g}_{\xi,\eta}}{w_0\tilde{g}_{\xi,\eta}} = \sum_j \left.\frac{w_1}{w_0}\right|_{p_j(\xi,\eta)} + R_\infty = \sum_j f_j(\xi,\eta) + R_\infty, \tag{5.10}$$

where R_∞ is the sum of the residues at $w_0 = 0$.

As $\int_\gamma \frac{w_1 dw_0}{w_0^2}$ is constant with respect to ξ, it suffices to show that $\frac{\partial^2}{\partial\xi^2}(R_\infty) = 0$. Consider a local irreducible portion of an analytic variety intersecting $w_0 = 0$. Such can be locally parameterized by $\lambda \mapsto (\lambda^n : w_1(\lambda) : w_2(\lambda))$, with $n \geq 1$, for λ small. The residue for this portion of local analytic variety at infinity is

$$\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial\lambda^{n-1}}\left.\left(w_1\frac{\frac{\partial}{\partial\lambda}(w_2-\eta w_1) - \xi n\lambda^{n-1}}{(w_2-\eta w_1)-\xi\lambda^n}\right)\right|_{\lambda=0}. \tag{5.11}$$

It is a basic exercise to see that this is ξ-affine.

□

Lemma 5.4. *Let V be a positive holomorphic 1-chain with finite mass bounded by γ within $\mathbb{CP}^2$ and suppose that V has no components in the line at infinity, $w_0 = 0$. For any domain $\Omega \subseteq \mathfrak{U}_\gamma\backslash\mathfrak{I}_V$, there exists a nonnegative integer N and holomorphic functions $e_1, e_2, \ldots, e_N$ satisfying the multi-shockwave equations (4.4) on Ω, such that*

$$\frac{\partial^2}{\partial\xi^2}G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial\xi^2}e_1(\xi,\eta) \tag{5.12}$$

Proof. As with Lemma 5.3, define N to be the degree of intersection between V and the line $w_2 = \xi w_0 + \eta w_1$. Define $e_k(\xi,\eta)$ on Ω to be the kth elementary symmetric function of the z_1 coordinates of the intersections, counting multiplicities, between V with the line $w_2 = \xi w_0 + \eta w_1$. By symmetry, the functions e_k are well-defined on Ω and are continuous since $\Omega \subseteq \mathfrak{U}_\gamma\backslash\mathfrak{I}_V$. By local application of Lemma 5.3 and Lemma 4.1, the functions $e_1, e_2, \ldots, e_N$ are holomorphic and satisfy the multi-shockwave system of equations and (5.12) on $\Omega\backslash\mathfrak{I}_V$. By a removable singularities argument, [16] (Lemma 3), these properties extend to all of Ω.

□

Lemma 5.5. *Let V be a positive holomorphic 1-chain with finite mass bounded by γ within $\mathbb{CP}^2$ and suppose that V has no components in the line at infinity, $w_0 = 0$. For any domain $\Omega \subseteq \mathfrak{U}_\gamma$, there exists a nonnegative integer N and holomorphic functions $P_0, P_1, \ldots, P_N$ satisfying the refined h.s.w. equations (4.10) on Ω, such that*

$$\frac{\partial^2}{\partial\xi^2}G_\gamma(\xi,\eta) = \frac{\partial^2}{\partial\xi^2}\frac{P_1(\xi,\eta)}{P_0(\xi,\eta)} \tag{5.13}$$

Proof. Define N and holomorphic functions $e_1, e_2, \dots, e_N$ on $\Omega\backslash\mathcal{I}_V$ as in Lemma 5.4. Let $\mathcal{E}_V$ be the finite set of all (ξ, η) such that V has a component contained in the line $w_2 = \xi w_0 + \eta w_1$.

Let $(\xi^*, \eta^*) \in \Omega \cap \mathcal{I}_V\backslash\mathcal{E}_V$, and define $q = (0 : 1 : \eta^*)$, which is a point of intersection between V and the line at infinity $w_0 = 0$. Let $u_1 = \frac{w_2 - \xi^* w_0 - \eta^* w_1}{w_1}$ and $u_2 = \frac{w_0}{w_1}$, which can be used as holomorphic coordinates near q. Let U be a neighborhood of q and let $F(u_1, u_2)$ be a holomorphic function on U such that the divisor of F is $V \cap U$. We may suppose that U has the form $\{|u_1| < \delta, |u_2| < \epsilon\}$, where δ and ϵ are chosen so that V does not intersect $\{|u_1| \le \delta, |u_2| = \epsilon\}$ and such that V intersects the lines $u_2 = 0$ and $u_1 = 0$ only at q. Define $\Omega_U = \Omega \cap \{(\xi, \eta) \mid \epsilon|\xi - \xi^*| + |\eta - \eta^*| < \delta\}$, shrinking δ if necessary to ensure that Ω_U is connected and disjoint from $\mathcal{E}_V$.

Let m be the degree of intersection between V and the line at infinity $u_2 = 0$ at q. In particular, m is the order of vanishing of $F(u_1, 0)$ at $u_1 = 0$.

Claim: There exists a constant C such that

$$|e_j(\xi, \eta)| \le \frac{C}{|\eta - \eta^*|^m}, \tag{5.14}$$

for $1 \le j \le N$ and $(\xi, \eta) \in \Omega_U\backslash\{\eta = \eta^*\}$.

For the moment, assume that this claim holds. Let $\eta_1^*, \dots, \eta_s^*$ be all the values of η^* for which $(0 : 1 : \eta^*)$ is a point of intersection between V and the line at infinity, and let $m_1, \dots, m_s$ give the corresponding degrees of intersection between V and $w_0 = 0$ at $(0 : 1 : \eta_j^*)$. Let $P_0(\xi, \eta) = \prod_{\ell=1}^{s}(\eta - \eta_\ell^*)^{m_\ell}$ and $P_k = P_0 e_k$ for $1 \le k \le N$, which give holomorphic functions on $\Omega\backslash\mathcal{I}_V$ that satisfy (5.13). By Lemma 4.2, we see that $P_0, \dots, P_N$ satisfy the refined h.s.w. system of equations on $\Omega\backslash\mathcal{I}_V$. Due to the claim above, each function P_k extends to a holomorphic function on $\Omega\backslash(\mathcal{E}_V \cap \mathcal{I}_V)$, which then extends holomorphically to Ω as $\mathcal{E}_V$ has codimension two. The appropriate properties likewise continue to all of Ω, thus establishing the lemma. So it only remains to prove the claim.

Let (ξ, η) be an arbitrary point in $\Omega_U\backslash\{\eta = \eta^*\}$. Let N_U denote the degree of intersection between V and the line $w_2 = \xi w_0 + \eta w_1$ inside U, which is constant for (ξ, η) in $\Omega_U\backslash\{\eta = \eta^*\}$. Let $e_{U,k}(\xi, \eta)$ be the kth elementary symmetric function of the z_1 (or $1/u_2$) coordinates of the intersections, counting multiplicity, between V and $w_2 = \xi w_0 + \eta w_1$ inside U. Let $c_{U,k}(\xi, \eta)$ be the sum of kth powers of the z_1 coordinates of the same intersections. Define the standard generating functions $E_U(t) = 1 + \sum_{k=1}^{N_U} e_{U,k} t^k$ and $C_U(t) = \sum_{k=1}^{\infty} c_{U,k} t^{k-1}$, which are related by the equations $C_U(-t) = \frac{E_U'(t)}{E_U(t)}$ and $E_U(t) = \exp\left(\int_0^t C_U(-\tau)\, d\tau\right)$ [14].

Let $H_{\xi,\eta}(\lambda) = F(((\xi-\xi^*)\lambda + (\eta-\eta^*),\lambda))$, which is non-vanishing on $|\lambda| = \epsilon$. For $k \geq 1$, define

$$S_k(\xi,\eta) = \frac{1}{2\pi \mathrm{i}} \int_{|\lambda|=\epsilon} \frac{1}{\lambda^k} \frac{H'_{\xi,\eta}(\lambda)}{H_{\xi,\eta}(\lambda)}\, d\lambda, \tag{5.15}$$

which is bounded for $(\xi,\eta) \in \Omega_U \backslash \{\eta = \eta^*\}$. By basic residue calculations,

$$c_{U,k}(\xi,\eta) = S_k(\xi,\eta) - \frac{1}{(k-1)!} \frac{d^{k-1}}{d\lambda^{k-1}} \left(\frac{H'_{\xi,\eta}(\lambda)}{H_{\xi,\eta}(\lambda)} \right)\Bigg|_{\lambda=0}. \tag{5.16}$$

So

$$C_U(t) = \sum_{k=1}^{\infty} S_k(\xi,\eta) t^{k-1} - \frac{H'_{\xi,\eta}(t)}{H_{\xi,\eta}(t)}, \tag{5.17}$$

and thus

$$E_U(t) = \exp\left(\sum_{k=1}^{\infty} \frac{-S_k(\xi,\eta)}{k} (-t)^k \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{H^{(k)}_{\xi,\eta}(0)}{H_{\xi,\eta}(0)} t^k. \tag{5.18}$$

Thus $e_{U,k}$ is a linear combination of the elements of $\left\{ \frac{H^{(j)}_{\xi,\eta}(0)}{H_{\xi,\eta}(0)} \right\}_{0\leq j\leq k}$ using coefficients in terms of $\{S_k(\xi,\eta)\}$. Since $H_{\xi,\eta}(0) = F(\eta-\eta^*,0)$ is comparable to $(\eta-\eta^*)^m$ and $H^{(j)}_{\xi,\eta}(0)$ is bounded for $(\xi,\eta) \in \Omega_U \backslash \{\eta = \eta^*\}$, it holds that there exists a constant C such that

$$|e_{U,k}(\xi,\eta)| \leq \frac{C}{|\eta-\eta^*|^m}, \tag{5.19}$$

for $1 \leq k \leq N_U$ and $(\xi,\eta) \in \Omega_U \backslash \{\eta = \eta^*\}$.

Finally, note that $e_j(\xi,\eta) = \sum_k a_{j-k}(\xi,\eta) e_{U,k}(\xi,\eta)$, where $a_\ell(\xi,\eta)$ is the ℓth elementary symmetric function, counting multiplicity, of the z_1 values of the intersections between V and the line $w_2 = \xi w_0 + \eta w_1$ outside U. By the estimate (5.19) and the fact that there are uniform bounds on the functions $a_\ell(\xi,\eta)$ for $(\xi,\eta) \in \Omega_U$, the claim follows.

□

Proof. (of Theorem 5.1). Assume that V is a holomorphic 1-chain bounded by γ within $\mathbb{CP}^2$. By separating positive and negative components, we may decompose V into the difference of two positive holomorphic 1-chains V^+ and V^-. Let $\gamma^+ = dV^+$ and $\gamma^- = dV^-$. Thus $V = V^+ - V^-$ and $\gamma = \gamma^+ - \gamma^-$.

Assuming (i) holds, then (iii') and (iv') follow as a result of Lemma 5.4 and Lemma 5.5, respectively, being applied to V^+ and V^-, with the former also requiring the fact that inclusion in $\mathcal{T}_V$ is only dependent on η for (ξ,η) in $\mathcal{U}_\gamma$. It is clear that (iv') implies (iv) and that (iii') implies (iii).

Assume (iv). Reposition (ξ^*,η^*) and shrink Ω so that P_0^+ and P_0^- are non-vanishing on Ω. By defining $e_k^+ = \frac{P_k^+}{P_0^+}$ and $e_k^- = \frac{P_k^-}{P_0^-}$, (iii) is satisfied, using Lemma 4.2

Assume (iii). Reposition (ξ^*, η^*) and shrink Ω so that $t^{N^+} + e_1^+ t^{N^+-1} + \cdots + e_{N^+}$ and $t^{N^-} + e_1^- t^{N^- -1} + \cdots + e_{N^-}$ split into a product of monic t-linear factors in the polynomial ring $\mathcal{O}(\Omega)[t]$. From the monic t-linear factors, each of which has the form $t + f(\xi, \eta)$, extract the t-constant terms to yield holomorphic functions $f_1^+, f_2^+, \ldots, f_{N^+}^+$ and $f_1^-, f_2^-, \ldots, f_{N^-}^-$ that satisfy (ii), owing to Lemma 4.1.

That (ii) implies (i) is shown in a paper by Dinh [3](Thèoréme 7.4).

□

6. Characterizing Decomposability

Theorem 5.1 and Corollary 5.2 characterize the closed rectifiable 1-currents that bound holomorphic 1-chains within $\mathbb{CP}^2$ by the existence of certain "shockwave decompositions" of G_γ. It is natural to inquire how one may determine when G_γ has such decompositions or not. This section focuses on this issue, culminating in Theorem 6.7 and the proofs of Theorem 1.1 and Theorem 1.2. We begin by presenting some introductory definitions and by outlining our pathway to the final theorems.

Let K be a non-empty, connected, compact set and let $G \in \mathcal{O}(K)$. We say that $P_0^+, \ldots, P_{N^+}^+ \in \mathcal{O}(K)$ and $P_0^-, \ldots, P_{N^-}^- \in \mathcal{O}(K)$, with P_0^+ and P_0^- not identically zero, represent a *h.s.w. decomposition* (with *signature* (N^+, N^-)) of G on K if both lists of germs satisfy the refined h.s.w. system of equations (4.10) and

$$\frac{\partial^2}{\partial \xi^2} G = \frac{\partial^2}{\partial \xi^2} \left(\frac{P_1^+}{P_0^+} - \frac{P_1^-}{P_0^-} \right). \tag{6.1}$$

We call $N^+ - N^-$ the *degree* of the decomposition, whereas we call $N^+ + N^-$ the *absolute degree* of the decomposition. A *positive h.s.w. decomposition* is a h.s.w. decomposition with signature $(N^+, 0)$, meaning that P_1^- be regarded as zero and $\frac{\partial^2}{\partial \xi^2} G = \frac{\partial^2}{\partial \xi^2} \left(\frac{P_1^+}{P_0^+} \right)$.

By Theorem 5.1 we see that γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$ if and only if G_γ has a h.s.w. decomposition for any arbitrarily selected non-empty, connected, compact set K in $\mathcal{U}_\gamma$.

Assumptions on K: For the remainder of this section we assume that K is a connected, Stein, compact set containing the point (ξ^*, η^*) such that

(a) $\mathcal{O}(K)$ is a unique factorization domain,
(b) K is Cauchy-viable with respect to $\xi = \xi^*$, and
(c) $\tilde{K} := \{\eta \in \mathbb{C} \mid (\xi^*, \eta) \in K\}$ is Cauchy-viable with respect to $\eta = \eta^*$.

(See Subsection 2.1 for definitions and information regarding Cauchy-viability.) These assumptions on K are stronger than those employed in Section 4, so the results and discussion of Section 4 apply. It may be instructive for the reader to keep the case where K is simply the set $\{(\xi^*, \eta^*)\}$ readily in mind, as it is informative and dovetails with Corollary 5.2.

Let μ be an element of $\mathcal{O}(K)$, and let N be a non-negative integer. We say that μ satisfies *condition* $(\star_N)$ if there exists $P_0, P_1, \ldots, P_N \in \mathcal{O}(K)$, with P_0 not identically zero, such that

$$(P_{k+1})_\xi = \mu P_k - (P_k)_\eta, \text{ for } 0 \leq k \leq N, \text{ and } \qquad (P_0)_\xi = 0. \tag{6.2}$$

where P_{N+1} is regarded as zero.

Positive h.s.w decompositions and condition $(\star_N)$ are directly connected, as is expressed in the proposition below.

Proposition 6.1. *Let N be a non-negative integer, and suppose that G and μ are elements of $\mathcal{O}(K)$ such that $\mu_\xi = G_{\xi\xi}$. G has a positive h.s.w. decomposition with degree N if and only if μ satisfies condition $(\star_N)$.*

Proof. From the discussion concluding Section 4, there is no loss of generality to suppose that $\mu|_{\xi=\xi^*} = 0$.

If G has a positive h.s.w. decomposition, then there exist $P_0, P_1, \ldots, P_N \in \mathcal{O}(K)$ satisfying (6.2) with some $\hat{\mu} \in \mathcal{O}(K)$ in place of μ. We may also assume, without loss of generality, that $\hat{\mu}|_{\xi=\xi^*} = 0$. Since $G_{\xi\xi} = \frac{\partial^2}{\partial \xi^2}\left(\frac{P_1}{P_0}\right) = \hat{\mu}_\xi$, it follows that $\hat{\mu} = \mu$.

If μ satisfies condition $(\star_N)$, then it follows that G has a positive h.s.w. decomposition with degree N owing to the definitions and the fact that (6.2) implies that $\mu_\xi = \frac{\partial^2}{\partial \xi^2}\left(\frac{P_1}{P_0}\right)$. □

Proposition 6.1 yields a subtle, yet significant, simplification. Identifying the existence of a positive h.s.w. decomposition requires finding suitable solutions to a system of *non-linear* first order partial differential equations, whereas verifying condition $(\star_N)$ involves finding solutions to a system of *linear* first order partial differential equations.

Also it is worthwhile to note that (6.2) is an overdetermined system of partial differential equations, consisting of $N+2$ equations on $N+1$ functions. Absent the equation $0 = \mu P_N - (P_N)_\eta$, (6.2) would yield an exactly determined initial value problem using D_ξ and taking Cauchy data on $\xi = \xi^*$. Similarly, removal of the equation $(P_0)_\xi = 0$, (6.2) would give an exactly determined initial value problem using D_η and taking Cauchy data on $\eta = \eta^*$.

In Subsection 6.1 we show that solutions to (6.2) correspond to the mutual solutions of a certain linear equation and two particular systems of exactly determined linear ordinary differential equations, one with respect to ξ and another with respect η. This permits us to apply the results of Section 3. Doing so, we synthesize our conclusions in Subsection 6.2, which shows, among other things, that condition

$(\star_N)$ on μ is equivalent to a finite set of explicitly constructible partial differential conditions on μ.

6.1. **An Ordinary Differential Representation of Condition $(\star_N)$.** The key result of this subsection is that the solutions to (6.2) can be characterized as the mutual solutions to a linear equation and two collections of ordinary differential equations. These results are ultimately expressed in Theorem 6.4 and Theorem 6.5, with an important identity expressed in Lemma 6.6.

We define the following objects for representing formal differential expressions. Let $\mathfrak{U}$ denote the free $\mathbb{Z}$-algebra with formal generators $\{\boldsymbol{D_\eta^i D_\xi^j \mu}\}_{i\geq 0, j\geq 1}$. For $n \geq 0$, let $\mathfrak{V}_n$ denote the free $\mathfrak{U}$-module generated by the formal elements $\{\boldsymbol{D_\xi^j P_i}\}_{0\leq j\leq i\leq n}$, and let $\mathfrak{V}_{-1}$ be the zero module.

Given $\mu \in \mathcal{O}(K)$ there is a uniquely defined ($\mathbb{Z}$-algebra) homomorphism from $\mathfrak{U}$ to $\mathcal{O}(K)$ defined by mapping each formal symbol $\boldsymbol{D_\eta^i D_\xi^j \mu}$ to $D_\eta^i D_\xi^j \mu$. Likewise, given $\mu, P_0, P_1, \ldots, P_n \in \mathcal{O}(n)$ there is a uniquely defined ($\mathbb{Z}$-module) homomorphism from $\mathfrak{V}_n$ to $\mathcal{O}(K)$ defined by evaluation. We will use Φ to denote the evaluation homomorphism appropriate to the given context.

[Notational note: To consolidate certain cases into fewer equations, we may employ, at times, the following notational devices. Let δ_a denote the Kronecker delta function, defined as 1 when $a = 0$ and 0 otherwise. Let $\binom{m}{n}$ denote the usual binomial coefficient when $0 \leq n \leq m$, but with the definition extended to be zero when $n < 0$ or $n > m$. A summation expression where the upper index is one less than the lower index, e.g. $\sum_{j=a}^{a-1} b_j$, is permitted, in which case it is simply treated as an empty sum and regarded as zero.]

We start with the following basic identity. Its proof is automatic.

Proposition 6.2. *Suppose that $\mu, P_0, P_1, \ldots, P_{N+1}$ satisfy (6.2). For $0 \leq m \leq N$, $n \geq 0$,*

$$\begin{aligned} D_\xi^{n+1} P_{m+1} &= D_\xi^n(\mu P_m - (P_m)_\eta) \\ &= \sum_{j=0}^{n-1} \left[\binom{n}{j} (D_\xi^{n-j}\mu)(D_\xi^j P_m) \right] + (\mu - D_\eta)(D_\xi^n P_m). \end{aligned} \tag{6.3}$$

The following lemma captures a remarkable feature of (6.2) and its solutions.

Lemma 6.3. *For $0 \leq k < \ell$, there exists an element $p_{k,\ell}$ of $\mathfrak{V}_{k-1}$ such that $D_\xi^\ell P_k = \Phi(p_{k,\ell})$ whenever $\mu, P_0, P_1, \ldots, P_{N+1} \in \mathcal{O}(K)$ satisfy (6.2) with $N \geq k-1$.*

Proof. First we prove the case $\ell = k+1$ by induction on k. This case is trivially true for $k = 0$, since $(P_0)_\xi = 0$. Also one may see that it is true for $k = 1$ since $(P_1)_{\xi\xi} = \frac{\partial}{\partial \xi}(\mu P_0 - (P_0)_\eta) = \mu_\xi P_0$.

Now assume that the statement holds for all non-negative k less than or equal to k', for a fixed $k' \geq 1$. By Proposition 6.2,

$$D_\xi^{k'+2} P_{k'+1} = \sum_{j=0}^{k'} \left[\binom{k'+1}{j} (D_\xi^{k'+1-j}\mu)(D_\xi^j P_{k'}) \right] + (\mu - D_\eta)\,(D_\xi^{k'+1} P_{k'}). \tag{6.4}$$

In the equation above, the summation term corresponds to a formal expression in $\mathfrak{V}_{k'}$, so we may simply direct our attention to the rightmost term. By the inductive hypothesis, this term agrees with a $\mathbb{Z}$-linear combination of terms of the form $(\mu - D_\eta)(rD_\xi^n P_m)$, where $r \in \mathfrak{U}$ and $0 \leq n \leq m \leq k'-1$. Observe that

$$\begin{aligned}(\mu - D_\eta)(rD_\xi^n P_m) &= r(\mu - D_\eta)(D_\xi^n P_m) - (D_\eta r) D_\xi^n P_m \\ &= rD_\xi^{n+1} P_{m+1} - \sum_{j=0}^{n-1} \left[\left(\binom{n}{j} r(D_\xi^{n-j}\mu) \right) (D_\xi^j P_m) \right] - (D_\eta r) D_\xi^n P_m,\end{aligned} \tag{6.5}$$

using Proposition 6.2 in the last step. Thus $(\mu - D_\eta)(rD_\xi^n P_m)$ agrees with an expression in $\mathfrak{V}_{k'}$. This proves the lemma's statement for $\ell = k+1$.

Define $\boldsymbol{D_\xi}$ as the map on $\mathfrak{V}_{k-1}$ that operates as formal differentiation on $\mathfrak{U}$ and satisfies $\boldsymbol{D_\xi}(\boldsymbol{D_\xi^j P_i}) = \boldsymbol{D_\xi^{j+1} P_i}$ for $0 \leq j < i \leq k-1$ and $\boldsymbol{D_\xi}(\boldsymbol{D_\xi^i P_i}) = p_{i,i+1}$ for $0 \leq i \leq k-1$. Thus this map agrees with D_ξ under evaluation, i.e. $\Phi \circ \boldsymbol{D_\xi} = D_\xi \circ \Phi$. Let $p_{k,\ell} = \boldsymbol{D_\xi}^{\ell-k-1}(p_{k,k+1})$, for $\ell > k+1$, which concludes the proof.

□

So Lemma 6.3 reveals that there exist $\rho_{k,\ell,i,j} \in \mathfrak{U}$ such that

$$D_\xi^\ell P_k = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \Phi(\rho_{k,\ell,i,j}) D_\xi^j P_i, \tag{6.6}$$

for $0 \leq k < \ell$, and for all $\mu, P_0, P_1, \ldots, P_{N+1} \in \mathcal{O}(K)$ that satisfy (6.2) with $N \geq k-1$. Furthermore, the proof of Lemma 6.3 expresses an explicit means for recursively constructing $\rho_{k,\ell,i,j}$. In the appendix, we use this to generate a constructive definition for $\rho_{k,\ell,i,j}$, which is stated in (A.1).

Our fundamental interest lies in the case $\ell = k+1$. So to shorten notation, we use $\nu_{k,i,j}$ to denote $\rho_{k,k+1,i,j}$. From the derived recursion formula for $\rho_{k,\ell,i,j}$ given in (A.1), it follows that we may define $\nu_{k,i,j}$ recursively as follows.

- For $0 \leq k$ and any of $j < 0$, $i < j$, or $i \geq k$,

$$\nu_{k,i,j} = 0.$$

- For $0 \le j \le i < k$,

$$\nu_{k,i,j} = -\sum_{j_1=j+1}^{i} \binom{j_1}{j} (D_\xi^{j_1-j}\mu)\nu_{k-1,i,j_1} - (D_\eta \nu_{k-1,i,j}) + \delta_{k-1-i}\binom{k}{j}(D_\xi^{k-j}\mu) + \nu_{k-1,i-1,j-1}. \tag{6.7}$$

Some examples of $p_{k,k+1}$, as derived from this definition, are listed below.

$$p_{1,2} = \boldsymbol{(D_\xi \mu) P_0}$$

$$p_{2,3} = -\boldsymbol{(D_\eta D_\xi \mu) P_0} + 3\boldsymbol{(D_\xi \mu)(D_\xi P_1)} + \boldsymbol{(D_\xi^2 \mu) P_1}$$

$$p_{3,4} = \boldsymbol{(D_\eta^2 D_\xi \mu) P_0} - 4\boldsymbol{(D_\eta D_\xi \mu)(D_\xi P_1)} - (3\boldsymbol{(D_\xi \mu)}^2 + \boldsymbol{(D_\eta D_\xi^2 \mu)})\boldsymbol{P_1} + 6\boldsymbol{(D_\xi \mu)(D_\xi^2 P_2)} + 4\boldsymbol{(D_\xi^2 \mu)(D_\xi P_2)} + \boldsymbol{(D_\xi^3 \mu) P_2}$$

For μ and $P_0, P_1, \ldots, P_N$ satisfying (6.2) with $P_{N+1} = 0$, the following equations are satisfied;

$$D_\xi(D_\xi^k P_k) = \sum_{i=0}^{k-1}\sum_{j=0}^{i} \Phi(\nu_{k,i,j}) D_\xi^j P_i, \text{ for } 0 \le k \le N, \tag{6.8}$$

$$0 = \sum_{i=0}^{N}\sum_{j=0}^{i} \Phi(\nu_{N+1,i,j}) D_\xi^j P_i, \tag{6.9}$$

and, due to Proposition 6.2,

$$D_\eta(D_\xi^j P_i) = \sum_{j'=0}^{j} \left[\binom{j}{j'}(D_\xi^{j-j'}\mu)(D_\xi^{j'} P_i)\right] - D_\xi^{j+1} P_{i+1}, \text{ for } 0 \le j \le i \le N, \tag{6.10}$$

where P_{N+1} is again regarded as zero.

The equations (6.8), (6.9), and (6.10) can be expressed in matrix form. (Recall the matrix and indexing protocol given in Subsection 2.2.) Let $I = \{(i,j) \mid 0 \le j \le i\}$ be an index set with the ordering $\prec$ defined such that $(i,j) \preceq (i'j')$ if and only if $i = i'$ and $j \ge j'$ or $i < i'$. In other words, $\prec$ is the lexicographical ordering of I, using a reverse ordering for the second entry. Let $I_N = \{(i,j) \mid 0 \le j \le i \le N\}$, which is a finite subset of I. Let

$$\vec{v}_N = \begin{bmatrix} v_{0,0} & v_{1,1} & v_{1,0} & v_{2,2} & \cdots & v_{2,0} & \cdots & v_{N,N} & \cdots & v_{N,0} \end{bmatrix}^{\mathrm{T}} \tag{6.11}$$

be a $I_N \times 1$ matrix with entries in $\mathcal{O}(K)$. The entry $v_{i,j}$ will serve to represent $D_\xi^j P_i$.

The equations of (6.8) can be expressed as $D_\xi \vec{v}_N = A_N \vec{v}_N$, where A_N is the $I_N \times I_N$ companion matrix with the following entry-wise definitions.

- For $(0,0) \preceq (i,j), (i',j') \preceq (N,0)$ with $i \ne j$, ${A_N}_{(i,j)}^{(i',j')} = \delta_{i-i'}\delta_{j+1-j'}$.

- For $(0,0) \preceq (i,i), (i',j') \preceq (N,0)$,
  ${A_N}_{(i,i)}^{(i',j')} = \nu_{i,i',j'}$ (which equals zero if $(i,i) \preceq (i',j')$).

Equation (6.9) can be expressed as $M_N \vec{v}_N = 0$, where M_N is the $1 \times I_N$ matrix with the entry-wise definition ${M_N}^{(i,j)} = \nu_{N+1,i,j}$.

The equations of (6.10) can be expressed in matrix form as $D_\eta \vec{v}_N = B_N \vec{v}_N$, where B_N is the $I_N \times I_N$ matrix defined entry-wise by the unified equation,

$$ {B_N}_{(i,j)}^{(i',j')} = \delta_{i-i'} \binom{j}{j'} D_\xi^{j-j'} \mu - \delta_{i+1-i'} \delta_{j+1-j'}. \tag{6.12} $$

Observations: The matrix A_N is strictly lower-triangular with respect to the ordering on I_N, and all of its entries are derived from expressions in $\mathfrak{U}$. The matrix B_N is upper triangular with respect to the ordering on I_N. The diagonal entries of B_N equal μ, while the entries above the diagonal are expressions in $\mathfrak{U}$.

Theorem 6.4. *If $P_0, P_1, \ldots, P_N$ satisfy (6.2) with $P_{N+1} = 0$ and $v_{i,j} = D_\xi^j P_i$ then $\vec{v}_N \in \ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)$.*

The proof simply follows from (6.8), (6.9), and (6.10). Also a stronger form of the converse holds.

Theorem 6.5. *If $\vec{v}_N \in \ker(D_\xi - A_N) \cap \ker(D_\eta - B_N)$ and $P_i = v_{i,0}$ for $0 \leq i \leq N$. Then $P_0, P_1, \ldots, P_N$ satisfy (6.2) with $P_{N+1} = 0$.*

Proof. Since $\vec{v}_N \in \ker(D_\xi - A_N)$, it holds that $D_\xi v_{0,0} = 0$ and $D_\xi v_{i,0} = v_{i,1}$ for $0 < i \leq N$. Since $\vec{v}_N \in \ker(D_\eta - B_N)$, it holds that $D_\eta v_{i,0} = \mu v_{i,0} - v_{i+1,1}$ for $0 \leq i < N$, and that $D_\eta v_{N,0} = \mu v_{N,0}$. Thus the functions $P_0 = v_{0,0}, P_1 = v_{1,0}, \ldots, P_N = v_{N,0}$ satisfy (6.2) with $P_{N+1} = 0$.

□

One key point, to be used later, is that $D_\xi - A_N$, $D_\eta - B_N$, and M_N satisfy the following commutator relationship.

Lemma 6.6. *Let Δ_N be the $I_N \times 1$ matrix defined entry-wise by ${\Delta_N}_{(i,j)} = \delta_{i-N} \delta_{j-N}$. Then*

$$ [D_\xi - A_N, D_\eta - B_N] = -\Delta_N M_N. \tag{6.13} $$

Proof. Expanding the commutator yields that

$$ [D_\xi - A_N, D_\eta - B_N] = (A_N)_\eta - (B_N)_\xi + A_N B_N - B_N A_N. \tag{6.14} $$

From (6.12), we calculate that

$$ (D_\xi B_N)_{(i,j)}^{(i',j')} = \delta_{i-i'} \binom{j}{j'} (D_\xi^{j-j'+1} \mu), \tag{6.15} $$

for general $(i, j,)$ and (i', j') in I_N.

To calculate the other terms of (6.14), we consider two separate cases. For the first case, we assume that $0 \le j < i \le N$. Then

$$(D_\eta A_N)^{(i',j')}_{(i,j)} = 0, \tag{6.16}$$

$$(A_N B_N)^{(i',j')}_{(i,j)} = B_N{}^{(i',j')}_{(i,j+1)} = \delta_{i-i'}\binom{j+1}{j'}(D_\xi^{j+1-j'}\mu) - \delta_{i+1-i'}\delta_{j+2-j'}, \tag{6.17}$$

and

$$\begin{aligned}(B_N A_N)^{(i',j')}_{(i,j)} &= \sum_{j_1=0}^{j}\binom{j}{j_1}(D_\xi^{j-j_1}\mu)A_N{}^{(i',j')}_{(i,j_1)} - \begin{cases} A_N{}^{(i',j')}_{(i+1,j+1)} & \text{if } i<N \\ 0 & \text{if } i=N \end{cases} \\ &= \delta_{i-i'}\binom{j}{j'-1}(D_\xi^{j-j'+1}\mu) - \delta_{i+1-i'}\delta_{j+2-j'}.\end{aligned} \tag{6.18}$$

Substituting these into (6.14) along with (6.15) yields that

$$\begin{aligned}&[D_\xi - A_N, D_\eta - B_N]^{(i',j')}_{(i,j)} \\ &\quad = \delta_{i-i'}\left[-\binom{j}{j'} + \binom{j+1}{j'} - \binom{j}{j'-1}\right](D_\xi^{j-j'+1}\mu) = 0,\end{aligned} \tag{6.19}$$

for $0 \le j < i \le N$ and $(i',j') \in I_N$.

For the second case, we assume that $0 \le j = i \le N$. Then

$$(D_\eta A_N)^{(i',j')}_{(i,i)} = D_\eta \nu_{i,i',j'}, \tag{6.20}$$

$$\begin{aligned}(A_N B_N)^{(i',j')}_{(i,i)} &= \sum_{i_1=0}^{N}\sum_{j_1=0}^{i_1}\nu_{i,i_1,j_1}\left(\delta_{i_1-i'}\binom{j_1}{j'}(D_\xi^{j_1-j'}\mu) - \delta_{i_1+1-i'}\delta_{j_1+1-j'}\right) \\ &= \sum_{j_1=j'+1}^{i'}\binom{j_1}{j'}(D_\xi^{j_1-j'}\mu)\nu_{i,i',j_1} + \mu\nu_{i,i',j'} - \nu_{i,i'-1,j'-1} \\ &= \mu\nu_{i,i',j'} - \nu_{i+1,i',j'} - D_\eta\nu_{i,i',j'} + \delta_{i-i'}\binom{i+1}{j'}(D_\xi^{i+1-j'}\mu),\end{aligned} \tag{6.21}$$

using the recursive definition (6.7) for $\nu_{i+1,i',j'}$ in the last equality, and

$$\begin{aligned}&(B_N A_N)^{(i',j')}_{(i,i)} \\ &= \sum_{i_1=0}^{N}\sum_{j_1=0}^{i_1-1}\left(\delta_{i-i_1}\binom{i}{j_1}(D_\xi^{i-j_1}\mu) - \delta_{i+1-i_1}\delta_{i+1-j_1}\right)\delta_{i_1-i'}\delta_{j_1+1-j'} \\ &\quad + \sum_{i_1=0}^{N}\left(\delta_{i-i_1}\binom{i}{i_1}(D_\xi^{i-i_1}\mu) - \delta_{i+1-i_1}\right)\nu_{i_1,i',j'} \\ &\quad = \delta_{i-i'}\binom{i}{j'-1}(D_\xi^{i-j'+1}\mu) + \mu\nu_{i,i',j'} - \begin{cases}\nu_{i+1,i',j'} & \text{if } i<N \\ 0 & \text{if } i=N\end{cases}.\end{aligned} \tag{6.22}$$

Substituting these into (6.14) along with (6.15) shows that

$$
\begin{aligned}
(6.23)\quad & [D_\xi - A_N, D_\eta - B_N]_{(i,i)}^{(i',j')} \\
&= \delta_{i-i'}\left[-\binom{i}{j'} + \binom{i+1}{j'} - \binom{i}{j'-1}\right](D_\xi^{i-j'+1}\mu) - \delta_{i-N}\nu_{N+1,i',j'} \\
&\qquad\qquad\qquad\qquad = -\delta_{i-N}{M_N}^{(i',j')},
\end{aligned}
$$

for $0 \leq j = i \leq N$ and $(i',j') \in I_N$.

□

6.2. **Concluding Synthesis.** Lemma 6.6 and the strict triangularity of A_N and $\tilde{B}_N$ (when $\mu_\xi|_{\xi=\xi^*} = 0$) place us in a situation where we can apply the results of Section 3 for determining when $\ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)$ is non-trivial.

For a $I_N \times 1$ matrix ψ with entries in $\mathcal{O}(K)$, let $T_N(\psi) = D_\xi(\psi) + \psi A_N$, $U_N(\psi) = D_\eta(\psi) + \psi B_N$, and $\check{U}_N(\psi) = D_\eta(\psi) + \psi(B_N - \mu\,\mathrm{Id}_{I_N})$.

Theorem 6.7. *Let N be a non-negative integer. Let K be a compact set containing a point (ξ^*, η^*) that satisfies the assumptions given at the beginning of Section 6, and suppose that $\mu \in \mathcal{O}(K)$. The function μ satisfies condition $(\star_N)$ if and only if $W_Y^{T_N,\check{U}_N}(M_N) = 0$ for every Young diagram (two-dimensional Young-like set) Y of cardinality $|I_N| = \binom{N+2}{2}$.*

Proof. The entries of $\check{U}_N^k(T_N^j(M_N))$ are differential expressions of μ_ξ, and thus so are the expressions produced by $W_Y^{T_N,\check{U}_N}(M_N)$. Due to this and the discussion concluding Section 4, it suffices to consider the case when $\mu|_{\xi=\xi^*} = 0$.

By Theorem 6.4 and Theorem 6.5 we have that μ satisfies condition $(\star_N)$ if and only if $\ker M_N \cap \ker(D_\xi - A_N) \cap \ker(D_\eta - B_N)$ is non-trivial. Note that A_N, B_N, and M_N satisfy the conditions for Corollary 3.9. So it follows that μ satisfies condition $(\star_N)$ if and only if $W_Y^{T_N,U_N}(M_N) = 0$ for every Young diagram Y of cardinality $|I_N|$. For any Young diagram Y, one may transform the matrix $M_Y^{T_N,U_N}(M_N)$ via elementary row operations into the matrix $M_Y^{T_N,\check{U}_N}(M_N)$, which implies that $W_Y^{T_N,\check{U}_N}(M_N) = W_Y^{T_N,U_N}(M_N)$.

□

Theorem 6.7 yields Theorem 1.2 as a special case. Now we prove Theorem 1.1.

Proof of Theorem 1.1. If γ bounds a holomorphic 1-chain V, then by separating the positive and negative components of V, we may define two positive holomorphic 1-chains V^+ and V^-, without any common components, such that $V = V^+ - V^-$. Then define $\gamma^+ = dV^+$ and $\gamma^- = dV^-$. Owing to the properties of condition A_1, such as those exhibited in [3](Section 1), if there exists an arc common to γ^+ and γ^- with the same orientation, then V^+ and V^- have components that locally agree somewhere, which is contrary to the construction of V^+ and V^-. By Lemma 5.5 and Proposition 6.1 we see that μ^+ and μ^-, as defined in the theorem, both satisfy condition $(\star_{N^+})$ and $(\star_{N^-})$, respectively, for large enough N^+ and N^-.

The converse result follows simply from Corollary 5.2 and Proposition 6.1. $\square$

Remarks:

(1) When determining whether γ bounds a holomorphic 1-chain that has only positive intersections with the line $z_2 = \xi^* + \eta^* z_1$, i.e. taking $N^- = 0$, it suffices to consider simply the decomposition $\gamma^+ = \gamma$ and $\gamma^- = 0$.

(2) A notable feature of Theorem 1.2 is that it characterizes whether μ statisfies condition $(\star_N)$ using purely differential conditions on μ_ξ. There exist other characterizations of condition $(\star_N)$ using integro-differential equations. We briefly outline here one related approach that was detailed in [18]. Let K_N be the fundamental matrix of $D_\xi - A_N$ normalized at $\xi = \xi^*$ and let L_N be the fundamental matrix of $D_\eta - \tilde{B}_N$ normalized at $\eta = \eta^*$, both of which can be constructed from A_N and $\tilde{B}_N$ by integration. Lemma 6.6 can be used to show that K_N and L_N provide changes of variables such that

$$\begin{aligned} L_N^{-1} K_N^{-1}(\ker(D_\xi - A_N) \cap \ker(D_\eta - B_N) \cap \ker(M_N)) \\ = \ker(D_\xi) \cap \ker(D_\eta) \cap \ker(M_N K_N L_N). \end{aligned} \tag{6.24}$$

Therefore $(\star_N)$ holds if and only if the entries of $M_N K_N L_N$ are linearly dependent. The entries of K_N and L_N can be expressed as integro-differential expressions of μ_ξ. And since linear dependence of holomorphic functions can be expressed in terms of generalized Wronskians, which are defined using differential operations, one can obtain an integro-differential characterization of functions satisfying condition $(\star_N)$ [19].

(3) An interesting application of the Dolbeault Henkin characterization within $\mathbb{CP}^2$ to the Inverse Dirichlet-Neumann problem on bordered Riemann surfaces is presented in [12]. That work also contains a characterization of the trace, i.e. e_1 in our terminology, of an algebroid multifunction solution to $f f_\xi = f_\eta$. Use of algebroid multifunctions eliminates some but not all of the genericity restraints on (ξ^*, η^*). (As in (iii') of Theorem 5.1, genericity of ξ^* can be eliminated, but genericity of η^* is retained.) The related characterization statement (Theorem 4 of [12]) is formulated in terms of the existence of holomorphic functions of η that satisfy a certain integro-differential equation.

(4) If γ is a simple, closed curve, then γ bounds a holomorphic 1-chain if and only if G_γ or $-G_\gamma$ satisfies $(\star_N)$ for some N. Example 3.2 of [11] yields something similar to this remark for $\mathbb{CP}^2 \backslash \mathbb{CP}^0$ via an approach using a

different arrangement of integral functions instead of G_γ. By Corollary 4.2 of [20], along with the Hadamard criteria for rationality as discussed in [21], one obtains a related result for $\mathbb{C} \times \mathbb{CP}^1$ that is expressible in differential terms.

(5) Consider the case where γ is a finite 1-chain with finitely many self-intersections. Let $\{\gamma_j\}$ denote the finite family of simple closed oriented curves whose orientation locally agrees with that of γ. There are only a finite number of ways that γ can be decomposed into $\gamma^+ - \gamma^-$ as described in Theorem 1.1, as γ^+ and $-\gamma^-$ must be positive linear combinations of the curves from $\{\gamma_j\}$. So, in this case, determining whether γ bounds a holomorphic 1-chain for a prescribed N^+ and N^- would involve only a finite number of partial differential equations on $\{G_{\gamma_j}\}$. (However, the number of ways that γ can be decomposed into $\gamma^+ - \gamma^-$ depends exponentially on the number of curves in $\{\gamma_j\}$.)

So Theorem 6.7 yields a finite set of explicitly calculable partial differential conditions that are together equivalent to condition $(\star_N)$. It may be possible to reduce the list of conditions. When considering general ϕ, the collection of Young-diagrams required for Corollary 3.9 cannot be reduced (See [19].) But the specifics of M_N, A_N, and B_N may limit the range of possible shapes for Y in Theorem 3.6 and so reduce the number of Young diagrams needed for Theorem 6.7.

To illustrate the previous comment, we discuss the case $N = 1$. With Theorem 6.7, we need no more than the following row matrices.

$$\begin{aligned}
M_1 &= [-\mu_{\xi\eta} \quad && 3\mu_\xi \quad && \mu_{\xi\xi} &&]\\
T_1(M_1) &= [-\mu_{\xi\xi\eta} + 3\mu_\xi\mu_\xi \quad && 4\mu_{\xi\xi} \quad && \mu_{\xi\xi\xi} &&]\\
T_1^2(M_1) &= [-\mu_{\xi\xi\xi\eta} + 10\mu_{\xi\xi}\mu_\xi \quad && 5\mu_{\xi\xi\xi} \quad && \mu_{\xi\xi\xi\xi} &&]\\
\check{U}_1(M_1) &= [-\mu_{\xi\eta\eta} \quad && 4\mu_{\xi\eta} \quad && \mu_{\xi\xi\eta} + 3\mu_\xi\mu_\xi &&]\\
\check{U}_1^2(M_1) &= [-\mu_{\xi\eta\eta\eta} \quad && 5\mu_{\xi\eta\eta} \quad && \mu_{\xi\xi\eta\eta} + 10\mu_{\xi\eta}\mu_\xi &&]
\end{aligned}$$

We claim that M_1 and $T_1(M_1)$ are linearly independent over $\mathcal{M}(K)$, unless $\mu_\xi = 0$. Suppose for the sake of contradiction that $\mu_\xi \neq 0$ and that M_1 and $T_1(M_1)$ are linearly dependent over $\mathcal{M}(K)$. Neither M_1 nor $T_1(M_1)$ is identically zero, so there exists a $k \in \mathcal{M}(K)\backslash\{0\}$ such that $T_1(M_1) = kM_1$. From this we may deduce that

$$0 = (D_\xi - \frac{1}{4}k)(4\mu_{\xi\xi} - 3k\mu_\xi) - 4(\mu_{\xi\xi\xi} - k\mu_{\xi\xi}) = (-\frac{1}{4}k^2 + k_\xi)(3\mu_\xi). \tag{6.25}$$

Therefore $k_\xi = \frac{1}{4}k^2$. Also we derive that

$$(6.26)\quad 0 = (D_\xi - k)(-\mu_{\xi\xi\eta} + 3\mu_\xi^2 + k\mu_{\xi\eta}) = -\mu_{\xi\xi\xi\eta} + 6\mu_{\xi\xi}\mu_\xi + 2k\mu_{\xi\xi\eta} - 3k\mu_\xi^2 - \frac{3}{4}k^2\mu_{\xi\eta},$$

and that

$$(6.27)\quad 0 = D_\eta(\mu_{\xi\xi\xi} - k\mu_{\xi\xi}) + \frac{1}{4}(D_\eta(k) - kD_\eta)(4\mu_{\xi\xi} - 3k\mu_\xi) = \mu_{\xi\xi\xi\eta} - 2k\mu_{\xi\xi\eta} + \frac{3}{4}k^2\mu_{\xi\eta}.$$

From these two equations we see that $0 = 6\mu_{\xi\xi}\mu_\xi - 3k\mu_\xi^2$. Since $4\mu_{\xi\xi} = 3k\mu_\xi$, it follows that $k\mu_\xi^2 = 0$, which yields a contradiction. Thus M_1 and $T_1(M_1)$ are linearly independent over $\mathcal{M}(K)$ if $\mu_\xi \neq 0$.

By the above, and owing to the construction of Y in Theorem 3.6, it follows that we need only to consider the Young diagrams of cardinality 3 that contain $(0,0)$ and $(0,1)$, namely $\{(0,0),(1,0),(2,0)\}$ and $\{(0,0),(1,0),(0,1)\}$. Consequentially, μ satisfies $(\star_1)$ if and only if

$$(6.28)\quad \begin{vmatrix} -\mu_{\xi\eta} & 3\mu_\xi & \mu_{\xi\xi} \\ -\mu_{\xi\xi\eta} + 3\mu_\xi\mu_\xi & 4\mu_{\xi\xi} & \mu_{\xi\xi\xi} \\ -\mu_{\xi\xi\xi\eta} + 10\mu_{\xi\xi}\mu_\xi & 5\mu_{\xi\xi\xi} & \mu_{\xi\xi\xi\xi} \end{vmatrix} = 0,$$

and

$$(6.29)\quad \begin{vmatrix} -\mu_{\xi\eta} & 3\mu_\xi & \mu_{\xi\xi} \\ -\mu_{\xi\xi\eta} + 3\mu_\xi\mu_\xi & 4\mu_{\xi\xi} & \mu_{\xi\xi\xi} \\ -\mu_{\xi\eta\eta} & 4\mu_{\xi\eta} & \mu_{\xi\xi\eta} + 3\mu_\xi\mu_\xi \end{vmatrix} = 0.$$

For general N, we conjecture that the row matrices $M_N, T_N(M_N), \ldots, T_N^N(M_N)$ are linearly independent over $\mathcal{M}(K)$, if $\mu_\xi \neq 0$. If this conjecture holds, then from the construction of Y in Theorem 3.6, it follows that we may adjust Theorem 6.7 to use only the Young diagrams of cardinality $|I_N|$ that contain $(0,0), (1,0), \ldots, (N,0)$.

Via a fairly lengthy calculation not presented here, the author has verified this conjecture for $N = 2$, i.e. the row matrices M_2, $T_2(M_2)$, and $T_N^2(M_2)$ are linearly independent over $\mathcal{M}(K)$ if $\mu_\xi \neq 0$. For $N = 2$, restricting to the Young diagrams of cardinality 6 that contain $(0,0)$, $(1,0)$, and $(2,0)$ would reduce the number of Young diagrams to be used from 11 to 7.

Appendix A. Derivation of Explicit Formulae for $p_{k,\ell}$

In this appendix we derive explicit recursion relations defining $p_{k,\ell}$ to satisfy Lemma 6.3 and we point out an identity that is a practical help in calculation.

We define $\rho_{k,\ell,i,j}$ recursively for $0 \leq k < \ell$ (and $i, j \in \mathbb{Z}$) as follows;

- For $0 \le k < \ell$, and for $j < 0$, $i < j$, or $i \ge k$,

$$\rho_{k,\ell,i,j} = 0.$$

- For $0 \le k < \ell$ and $0 \le j \le i < k$,

$$\begin{aligned}(\text{A.1})\quad & \rho_{k,\ell,i,j} = \\ & \sum_{j_1=k}^{\ell-2} \binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)\rho_{k-1,j_1,i,j} - \sum_{j_1=j+1}^{i} \binom{j_1}{j} (D_\xi^{j_1-j}\mu)\rho_{k-1,\ell-1,i,j_1} \\ & \qquad + \delta_{k-1-i}\binom{\ell-1}{j}(D_\xi^{\ell-1-j}\mu) - (D_\eta \rho_{k-1,\ell-1,i,j}) + \rho_{k-1,\ell-1,i-1,j-1}.\end{aligned}$$

(One can verify that the above constitutes a valid recursive definition by induction on k.)

Theorem A.1. *Lemma 6.3 is satisfied by defining*

$$p_{k,\ell} = \sum_{i=0}^{k-1}\sum_{j=0}^{i} \rho_{k,\ell,i,j} \boldsymbol{D_\xi^j P_i}, \tag{A.2}$$

for $0 \le k < \ell$.

Proof. Assume that $\mu, P_0, P_1, \ldots, P_{N+1}$ are general functions satisfying (6.2) for $N \ge k-1$. We prove the theorem by induction on k. Note the the theorem holds clearly in the case $k = 0$ (since $(P_0)_\xi = 0$).

Assume that the theorem holds with k replaced by k' for a k' such that $0 \le k' < k$. By Proposition 6.2,

$$\begin{aligned} D_\xi^\ell P_k = \sum_{j_1=k}^{\ell-2} & \left[\binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)(D_\xi^{j_1} P_{k-1}) \right] \\ & + \sum_{j_1=0}^{k-1} \left[\binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu)(D_\xi^{j_1} P_{k-1}) \right] + (\mu - D_\eta)(D_\xi^{\ell-1} P_{k-1}),\end{aligned}$$

which equals, by employing the inductive hypothesis,

$$\begin{aligned} \sum_{j_1=k}^{\ell-2} & \left[\binom{\ell-1}{j_1} (D_\xi^{\ell-1-j_1}\mu) \sum_{i=0}^{k-2}\sum_{j=0}^{i} \rho_{k-1,j_1,i,j}(D_\xi^j P_i) \right] \\ & \qquad + \sum_{i=0}^{k-1}\sum_{j=0}^{i} \left[\delta_{k-1-i} \binom{\ell-1}{j} (D_\xi^{\ell-1-j}\mu)(D_\xi^j P_i) \right] \\ & + \sum_{i=0}^{k-2}\sum_{j=0}^{i} \left[-(D_\eta \rho_{k-1,\ell-1,i,j})(D_\xi^j P_i) + \rho_{k-1,\ell-1,i,j} \Big((\mu - D_\eta) D_\xi^j P_i \Big) \right].\end{aligned}$$

By regrouping terms and using Proposition 6.2, this is equal to

$$\sum_{i=0}^{k-1}\sum_{j=0}^{i}\Bigg[\Bigg(\sum_{j_1=k}^{\ell-2}\binom{\ell-1}{j_1}(D_\xi^{\ell-1-j_1}\mu)\rho_{k-1,j_1,i,j} + \delta_{k-1-i}\binom{\ell-1}{j}(D_\xi^{\ell-1-j}\mu) - (D_\eta\rho_{k-1,\ell-1,i,j})\Bigg)D_\xi^j P_i\Bigg] + \sum_{i=0}^{k-2}\sum_{j_1=0}^{i}\Bigg[\rho_{k-1,\ell-1,i,j_1}\Bigg(D_\xi^{j_1+1}P_{i+1} - \sum_{j=0}^{j_1-1}\binom{j_1}{j}(D_\xi^{j_1-j}\mu)(D_\xi^j P_i)\Bigg)\Bigg],$$

producing, by interchanging or re-indexing summation operations,

$$\sum_{i=0}^{k-1}\sum_{j=0}^{i}\Bigg[\Bigg(\sum_{j_1=k}^{\ell-2}\binom{\ell-1}{j_1}(D_\xi^{\ell-1-j_1}\mu)\rho_{k-1,j_1,i,j} + \delta_{k-1-i}\binom{\ell-1}{j}(D_\xi^{\ell-1-j}\mu) - (D_\eta\rho_{k-1,\ell-1,i,j}) + \rho_{k-1,\ell-1,i-1,j_1-1} - \sum_{j_1=j+1}^{i}\rho_{k-1,\ell-1,i,j_1}\binom{j_1}{j}(D_\xi^{j_1-j}\mu)\Bigg)D_\xi^j P_i\Bigg],$$

which equals, by the definition (A.1) of $\rho_{k,\ell,i,j}$,

$$\sum_{i=0}^{k-1}\sum_{j=0}^{i}\Big[\rho_{k,\ell,i,j}D_\xi^j P_i\Big]. \tag{A.3}$$

□

Remarks: The definition of $p_{k,\ell}$ according to (A.2) agrees with the approach used the proof of Lemma 6.3. In the case that $\ell = k+1$, the proof of Theorem A.1 is simply the proof of Lemma 6.3 plus "bookkeeping". For $\ell > k+1$ it also holds that $p_{k,\ell}$ equals $\boldsymbol{D_\xi}(p_{k,\ell-1})$, according to the definition of $\boldsymbol{D_\xi}$ in the proof of Lemma 6.3. However a proof of this requires showing the identity $\rho_{k,\ell+1,i,j} = D_\xi(\rho_{k,\ell,i,j}) + \rho_{k,\ell,i,j-1} + \sum_{i'=i+1}^{k-1}\rho_{k,\ell,i',i'}\rho_{i',i'+1,i,j}$ for $0 \le j \le i < k < \ell$, which requires a rather arduous calculation that we omit here.

We also mention the following identity, which can help simplify calculations of $\rho_{k,\ell,i,j}$.

Theorem A.2. *For $j \ge 0$ and $0 \le k < \ell$,*

$$\rho_{k,\ell,i,j} = \binom{\ell}{j}\rho_{k-j,\ell-j,i-j,0}. \tag{A.4}$$

Proof. If $k \le i$ or $i < j$, the identity holds trivially. So we may suppose that $0 \le j \le i < k < \ell$. One may calculate that

$$\rho_{i+1,\ell,i,j} = \sum_{j_2=0}^{j}\binom{\ell-j_2-1}{j-j_2}(D_\xi^{\ell-1-j}\mu) = \binom{\ell}{j}D_\xi^{\ell-1-j}\mu, \tag{A.5}$$

which is sufficient to prove the identity in the case $k = i + 1$.

We proceed by induction on $k-i$. Assume that the identity holds when $k-i < m$ for some $m \geq 2$ and suppose that $k-i = m$. (What remains is a technical calculation that we simply summarize.) Apply the definition (A.2) recursively ($j+1$ times) to its left-most term $\rho_{k-1,\ell-1,i-1,j-1}$ to generate a formula for $\rho_{k,\ell,i,j}$ where all of the terms involved can employ the inductive hypothesis. By broad application of the inductive hypothesis, substantial use of the properties of binomial coefficients, and some summation manipulation, the identity follows.

□

Acknowledgments.

This material is based in part upon work for the author's doctoral dissertation at the University of Michigan and work while the author was a Ross Visiting Assistant Professor at the the Ohio State University. The author would like to thank David Barrett for his guidance as a thesis advisor and for thoughtful discussions pertaining to this material.

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