Additional Homework Exercises on Duality and Complementary Basic Solutions

A.

Consider the following linear programming problem as our primal problem.

Maximize \( Z = 2x_1 + 5x_2 + 3x_3 \)

subject to

\[
\begin{align*}
2x_1 + x_2 + 2x_3 &\leq 10 \\
x_1 + 6x_2 + 2x_3 &\leq 18
\end{align*}
\]

and

\[
\begin{align*}
x_1 &\geq 0, \\
x_2 &\geq 0, \\
x_3 &\geq 0
\end{align*}
\]

(a) Construct the dual problem to the above.

(b) Let \( s_1 \) and \( s_2 \) denote the slack variables for the first and second constraints (to the primal problem). Consider the CPF \((x_1, x_2, x_3, s_1, s_2) = (0, 3, 0, 7, 0)\). Find the complementary basic solution for the dual problem. (Remember: You will need to use the Complementary Slackness Property to start to identify basic and non-basic variables in the dual solution.) (Note: Let \( y_1, y_2, z_1, z_2, z_3 \) be the dual variables, with \( y_1 \) and \( y_2 \) being the dual decision variables and \( z_1, z_2, z_3 \) being the surplus variables arising in the dual constraints.)

(c) Now consider the corner point solution \((x_1, x_2, x_3) = (4, 2, 0)\). What is the associated basic solution \((x_1, x_2, s_1, s_2, s_3, s_4)\)? What is the complementary basic solution for the dual problem? Is this corner point solution optimal, suboptimal, superoptimal, or none of the above?

(d) Perform part (c) for the corner point solution \((x_1, x_2, x_3) = (0, 4, 3)\).

B.

Consider the following linear programming problem as our primal problem.

Minimize \( Z = 3x_1 + 2x_2 \)

subject to

\[
\begin{align*}
2x_1 + x_2 &\geq 18 \\
x_1 + x_2 &\geq 14 \\
x_1 + 4x_2 &\geq 32 \\
x_2 &\geq 3
\end{align*}
\]

and

\[
\begin{align*}
x_1 &\geq 0, \\
x_2 &\geq 0
\end{align*}
\]

(a) Construct the dual problem to the above.

(b) Let \( S_1, S_2, S_3, S_4 \) denote the surplus variables for the four constraints to the primal problem, respectively. Find the complementary basic solution to the basic solution \((x_1, x_2, S_1, S_2, S_3, S_4) = (0, 3, 15, 11, 20, 0)\). (Use variables of the form \( y_j \) to denote the usual dual decision variables and use variables of the form \( z_j \) to denote any augmented dual variables.) Is the primal basic solution optimal, suboptimal, superoptimal, or none of the above?

(c) Now consider the corner point solution \((x_1, x_2) = (4, 10)\). What is the associated basic solution \((x_1, x_2, S_1, S_2, S_3, S_4)\)? What is the complementary basic solution for the dual problem? Is this corner point solution optimal, suboptimal, superoptimal, or none of the above?