CHARACTERIZATIONS OF BOUNDARIES OF HOLOMORPHIC 1-CHAINS WITHIN $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ AND $\mathbb{C} \times \hat{\mathbb{C}}$

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Abstract. By way of a particular Cauchy-type integral, we characterize the closed rectifiable 1-currents γ, with support contained in $\mathbb{C}^2$ and satisfying condition A_1, that bound holomorphic 1-chains within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. And we derive characterizations for the boundaries of holomorphic 1-chains within $\mathbb{C} \times \hat{\mathbb{C}}$, which yield examples of characterizations within a non-compact, non-Stein space. Also we illustrate a connection between some of these characterizations and the Cauchy integral characterization of boundary values of meromorphic and holomorphic functions over a domain in $\mathbb{C}$.

1. Introduction

One topic of interest is the characterization of the real curves that are boundaries of analytic varieties of complex dimension 1. A generalization of this is the characterization of the closed rectifiable 1-currents that are boundaries of holomorphic 1-chains. The set of boundaries of holomorphic 1-chains is a biholomorphic invariant of the ambient space.

Wermer [14] considered the case of a simple closed curve in $\mathbb{C}^n$ that possessed an analytic parameterization with the first coordinate providing an immersion into $\mathbb{C}$.

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Wermer showed that such a curve bounds an analytic variety within $\mathbb{C}^n$ if and only if the curve has a non-trivial polynomial hull, and furthermore demonstrated that this is equivalent to the vanishing of a class of monomial moments of the curve. Work by Harvey and Lawson [8] (Theorem 7.2 and Remark 7.3) show that a closed, oriented, C^2 1-chain γ with disjoint components bounds a holomorphic 1-chain within a Stein ambient space if and only if $\int_\gamma \alpha = 0$ for all holomorphic 1-forms α of the ambient space. The condition on γ that $\int_\gamma \alpha = 0$ for all holomorphic 1-forms α is frequently referred to as the *moment condition*.

Regarding non-Stein spaces, work of Dolbeault and Henkin [3] provides one characterization within the ambient space $\mathbb{CP}^n$. (Work by El Kasimi [5] also provides a related characterization for the boundaries of holomorphic 1-chains within Grassmanians.) The proof within $\mathbb{CP}^n$ is performed by a reduction, via projections, to the case of $\mathbb{CP}^2$, which contains the essential spirit of the general result. For this reason, this article directs its focus on ambient spaces with complex dimension 2.

The characterization of boundaries of holomorphic 1-chains within $\mathbb{CP}^2$ due to Dolbeault and Henkin can be summarized as follows. Define the meromorphic 1-form $\omega = \frac{1}{2\pi \mathrm{i}} z_1 \frac{d(z_2 - \eta z_1)}{z_2 - \xi - \eta z_1}$ on $\mathbb{C}^2$, with coordinates (z_1, z_2) and parameters ξ and η, and let γ be a closed, oriented, C^2 1-chain in $\mathbb{C}^2 \subset \mathbb{CP}^2$. Then γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$ if and only if there is some point (ξ^*, η^*) about which the integral $\int_\gamma \omega$, which produces a function of ξ and η, may be locally decomposed into a signed sum of holomorphic solutions to the sign-inverted form of Burger's shockwave equation, $f f_\xi = f_\eta$. By examination of their subsequent work [4], the existence of such a decomposition modulo ξ-affine terms is also equivalent.

The regularity of the 1-chains being considered can be weakened. Dinh demonstrates that the Harvey and Lawson result in $\mathbb{C}^n$ and the Dolbeault and Henkin result in $\mathbb{CP}^n$ are valid for rectifiable 1-currents γ whose supports satisfy a condition termed A_1 [2].

We formulate a characterization for boundaries of holomorphic 1-chains within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ that does not require a holomorphic shockwave decomposition. (There is the natural birational map between $\mathbb{CP}^2$ and $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ arising from the identity map on $\mathbb{C}^2$. So, if γ is a rectifiable 1-current contained in $\mathbb{C}^2$, then it bounds an holomorphic 1-chain within $\mathbb{CP}^2$ if and only if γ does so within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.) In place of ω, we consider the form $\nu = \frac{1}{2\pi i} \frac{1}{\xi - w} \frac{dz}{z - \zeta}$ on $\mathbb{C}^2$ with coordinates (z, w) and parameters ζ and ξ. We show that a rectifiable 1-current γ satisfying A_1 bounds a holomorphic 1-chain within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ if and only if the integral $\int_\gamma \nu$ may be decomposed (either locally or globally) into a sum of ξ-logarithmic derivatives of a pair of functions, one being rational with respect to ζ and the other being rational with respect to ξ. This characterization is the subject of Section 3 and is stated in Theorem 3.1, with supporting definitions given in Section 2.

This characterization helps address the natural question as to whether shockwave decompositions are an essential part in characterizing the boundaries of holomorphic 1-chains in projective space. Also this characterization produces a condition that may be applied about any point (ζ^*, ξ^*) where the integral of ν is defined and continuous. (In contrast, when γ bounds a holomorphic 1-chain within $\mathbb{CP}^2$ (or $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$) there may exist a nonempty set of exceptional points (ξ^*, η^*) about which $\int \omega$ is defined and continuous yet the integral fails to have a proper shockwave decomposition.)

One feature of this characterization is that it naturally leads to a characterization within $\mathbb{C} \times \hat{\mathbb{C}}$, which is discussed in Section 4. This yields an example of a characterization of boundaries of holomorphic 1-chains within a non-Stein, non-compact space. (Past characterizations of boundaries of holomorphic 1-chains appear to occur within either Stein or compact spaces, though there exist some characterizations of boundaries of holomorphic p-chains within certain non-compact, non-Stein spaces for $p > 1$ [9], [4].)

Then we conclude in Section 5 with some remarks about some connections between these characterizations and the Cauchy integral characterizations of the boundary values of holomorphic and meromorphic functions over domains in $\mathbb{C}$.

2. Preliminaries

2.1. **Rectifiable Currents, Holomorphic Chains, and Boundaries of Holomorphic 1-Chains.** Much of the background discussed here involves the language of currents and geometric measure theory. For an extensive treatment of the subject of geometric measure theory, see Federer [6]. For an introduction devoted to the techniques and definitions particularly pertinent to holomorphic chains and their boundaries, one is encouraged to view the article by Harvey [7].

Let X be a complex manifold of complex dimension n with some Hermitian metric provided. The given Hermitian metric is used to define k-dimensional Hausdorff measure $\mathcal{H}^k$ on X and the mass seminorms for currents on X.

Let $E \subseteq X$. For $m \geq 1$, we say that E is m-rectifiable if and only if E is contained in the image of a Lipschitzian map from a bounded set in $\mathbb{R}^m$ to X. (E is said to be 0-rectifiable if and only if it is a finite set.) We say that E is

$(\mathcal{H}^m, m)$-rectifiable if and only if $\mathcal{H}^m(E) < \infty$ and for every $\epsilon > 0$ there exists an m-rectifiable set F such that $\mathcal{H}^m(E \backslash F) < \epsilon$.

$(\mathcal{H}^m, m)$-rectifiable sets generalize C^1 submanifolds of X in certain respects. (In fact, E is $(\mathcal{H}^m, m)$-rectifiable if and only if $\mathcal{H}^m(E) < \infty$ and $\mathcal{H}^m$-almost all of E is contained in some countable union of C^1 submanifolds of X, [6] (3.2.29).) A $(\mathcal{H}^m, m)$-rectifiable set E possesses an approximate tangent cone, denoted $\operatorname{Tan}^m(\mathcal{H}^m \llcorner E, x)$, which is a real m-dimensional vector space for $\mathcal{H}^m$-almost all $x \in E$ [6] (3.2.19). However the same cannot generally be said in regards to the usual tangent cone, denoted $\operatorname{Tan}(E, x)$. (See Federer [6] (3.1.21, 3.2.16) or Dinh [2] for the definitions of tangent cones and approximate tangent cones.) If a set E is $(\mathcal{H}^m, m)$-rectifiable and $\operatorname{Tan}(E, x)$ is a real m-dimensional vector space for $\mathcal{H}^m$-almost all $x \in E$, then we say E satisfies condition A_m [2].

The notion of rectifiability can be defined for currents in a number of equivalent ways [6](4.1.24, 4.1.28), [7](Appendix to Section 1). As one definition, a m-current (a current of dimension m) is called rectifiable if it has compact support and may be approximated by (integral) Lipschitzian m-chains in the mass topology. Equivalently, a m-current S is rectifiable if and only if $\operatorname{spt} S$ is compact and there exists a $(\mathcal{H}^m, m)$-rectifiable set $B \subseteq \operatorname{spt} S$ and a $(\mathcal{H}^m \llcorner B)$-integrable m-vector field η defined on B, where $\eta(x)$ is an integer multiple of one of the two decomposable m-vectors representing $\operatorname{Tan}^m(\mathcal{H}^m \llcorner B, x)$ for $\mathcal{H}^m$-almost all $x \in B$, such that $S = (\mathcal{H}^m \llcorner B) \wedge \eta$, meaning that

$$S(\phi) = \int_B \langle \eta, \phi \rangle \, d\mathcal{H}^m \tag{1}$$

for all test forms ϕ of degree m on X. We will customarily use the notation $\int_S \phi$ to represent the integral in (1) for any C^∞ m-form ϕ and any rectifiable m-current S.

An analytic variety V of pure dimension p in X possesses a current of integration, denoted $[V]$, defined by $[V](\phi) = \int_{\operatorname{Reg} V} \phi$ for test forms ϕ of degree p on X, where $\operatorname{Reg} V$ denotes the regular part of V [12]. A holomorphic p-chain T in X can be viewed as a locally finite formal sum of analytic varieties of pure dimension p in X with integer coefficients, usually represented as $\sum n_j V_j$, where $\{V_j\}$ are irreducible components of an analytic variety of pure dimension p and n_j is the associated integer multiplicity for V_j. A holomorphic p-chain $T = \sum n_j V_j$ has a corresponding current of integration $[T] = \sum n_j [V_j]$ in X. However a holomorphic p-chain is customarily identified with its current of integration, usually without making a notational distinction. (In fact, in several places in the literature, a holomorphic p-chain is defined solely as a p-current in the fashion of $[T]$ without any preceding formal object.)

Let γ be a closed 1-current in X, and let T be a holomorphic 1-chain in $X \setminus \operatorname{spt} \gamma$ with a simple extension to a current in X, which we will also denote by T. We say that γ *bounds* T (or that γ is the *boundary* of T) *within* X if both of the following conditions hold:

- $dT = \gamma$ as currents in X, and
- $\operatorname{spt} T \Subset X$.

A current of order zero S in $X \setminus \operatorname{spt} \gamma$ has a simple extension to a current in X if and only if for each $p \in \operatorname{spt} \gamma$ there exist a neighborhood U in X such that the mass of S in $U \setminus \operatorname{spt} \gamma$ is finite [12]. In particular, having finite (total) mass is a sufficient condition for having a simple extension. Conversely if T is holomorphic 1-chain in $X \setminus \operatorname{spt} \gamma$ that has a simple extension to X and is bounded by γ within X, then T has finite mass with respect to any Hermitian metric on X.

Some notational remarks regarding the statement "γ bounds a holomorphic 1-chain T within X":

- We routinely refer to X as the *ambient space*. Recognizing the ambient space is usually an important part of the denotation. However in cases where the ambient space can be clearly understood from the context, we may drop the phrase "within X".
- Unless $\gamma = 0$, T is not a holomorphic 1-chain in X. So references to T as a 1-holomorphic chain implicitly recognize T as a current in X that corresponds to a holomorphic 1-chain in $X \setminus \operatorname{spt} \gamma$.

2.2. **Rational Functions.** For a point $(\zeta^*, \xi^*) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ with coordinates (ζ, ξ) for $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, define $\mathcal{O}_{(\zeta^*,\xi^*)}$ as the ring of germs of holomorphic functions about (ζ^*, ξ^*), $\mathcal{O}^*_{(\zeta^*,\xi^*)}$ as the group of germs of non-vanishing holomorphic functions about (ζ^*, ξ^*), and $\mathcal{M}_{(\zeta^*,\xi^*)}$ as the field of germs of meromorphic functions about (ζ^*, ξ^*). Similarly define $\mathcal{O}_{\zeta^*}$, $\mathcal{O}^*_{\zeta^*}$, and $\mathcal{M}_{\zeta^*}$ for $\zeta^* \in \hat{\mathbb{C}}$ with coordinate ζ.

Define $\mathrm{RAT}^{\xi}_{(\zeta^*,\xi^*)}$ to be the fraction field of the polynomial ring $\mathcal{M}_{\zeta^*}[\xi]$, which has a natural inclusion as a subfield of $\mathcal{M}_{(\zeta^*,\xi^*)}$. So the elements of $\mathrm{RAT}^{\xi}_{(\zeta^*,\xi^*)}$ are meromorphic germs that may be expressed in the form

$$\frac{a_m(\zeta)\xi^m + a_{m-1}(\zeta)\xi^{m-1} + \cdots + a_0(\zeta)}{b_n(\zeta)\xi^n + b_{n-1}(\zeta)\xi^{n-1} + \cdots + b_0(\zeta)}, \tag{2}$$

for some non-negative integers m and n and germs of functions $a_m, \ldots, a_0$ and $b_n, \ldots, b_0$ contained in $\mathcal{M}_{\zeta^*}$, with b_n not identically zero. By clearing the denominators of the coefficients, there is no loss of generality to impose that the coefficients $a_m, \ldots, a_0$ and $b_n, \ldots, b_0$ be chosen from $\mathcal{O}_{\zeta^*}$. The elements of $\mathrm{RAT}^{\xi}_{(\zeta^*,\xi^*)}$ are called the *germs of functions rational with respect to* ξ *about* (ζ^*, ξ^*). Define $\mathrm{RAT}^{\zeta}_{(\zeta^*,\xi^*)}$ analogously.

Let Ω be a domain in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Define $\mathcal{O}(\Omega)$ as the ring of holomorphic functions on Ω, $\mathcal{O}^*(\Omega)$ as the group of non-vanishing holomorphic functions on Ω, and $\mathcal{M}(\Omega)$ as the field of meromorphic functions on Ω. The definitions of the objects $\mathcal{O}(U)$, $\mathcal{O}^*(U)$, and $\mathcal{M}(U)$ for a domain $U \subseteq \hat{\mathbb{C}}$ are similarly understood.

We will typically consider Ω to be a product domain of the form $U_1 \times U_2$. Define $\mathrm{RAT}^{\xi}(U_1 \times U_2)$ to be the fraction field of the polynomial ring $\mathcal{M}(U_1)[\xi]$, with a natural inclusion in $\mathcal{M}(U_1 \times U_2)$. The representation in (2) also applies to functions in $\mathrm{RAT}^{\xi}(U_1 \times U_2)$, using coefficients in $\mathcal{M}(U_1)$. The imposition of holomorphic coefficients, i.e. coefficients in $\mathcal{O}(U_1)$, in this representation is possible if U_1 is a proper subset of $\hat{\mathbb{C}}$. The elements of $\mathrm{RAT}^{\xi}(U_1 \times U_2)$ are called the *functions rational with respect to* ξ *on* $U_1 \times U_2$. Define $\mathrm{RAT}^{\zeta}(U_1 \times U_2)$ in an analogous fashion.

3. A Characterization within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$

For $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ we will use the bi-homogeneous coordinates $(z_0 : z_1) \times (w_0 : w_1)$ and the associated affine coordinates $(z, w) = (\frac{z_1}{z_0}, \frac{w_1}{w_0})$. Define π_1 and π_2 as the respective coordinate projections from $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$. We consider the product metric of the spherical metric on each $\hat{\mathbb{C}}$ as the standard metric for $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.

Let γ be a closed, rectifiable 1-current with support contained in $\mathbb{C}^2$. Define $\nu = \frac{1}{\xi - w} \frac{dz}{z - \zeta} = \frac{w_0}{\xi w_0 - w_1} \left(\frac{z_0 dz_1 - z_1 dz_0}{z_0(z_1 - \zeta z_0)} \right)$ as a (ξ, ζ) parameterized meromorphic 1-form in $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Define the function H_γ by

$$H_\gamma(\zeta, \xi) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \nu = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{\xi - w} \frac{dz}{z - \zeta}. \tag{3}$$

Define $\mathcal{U}^\zeta_\gamma = \hat{\mathbb{C}} \backslash \pi_1(\operatorname{spt}\gamma)$ and $\mathcal{U}^\xi_\gamma = \hat{\mathbb{C}} \backslash \pi_2(\operatorname{spt}\gamma)$. Note that $H_\gamma(\zeta, \xi)$ is well-defined and holomorphic on $\mathcal{U}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$, vanishing when $\zeta = \infty$ or $\xi = \infty$.

Theorem 3.1. *Let γ be a closed, rectifiable 1-current and suppose that* $\operatorname{spt}\gamma$ *is contained in $\mathbb{C}^2$ and satisfies condition A_1. The following are equivalent:*

(1) *γ bounds a holomorphic 1-chain, with finite mass, within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$.*

(2) *For any point (ζ^*, ξ^*) with any connected neighborhood $\Omega = U_1 \times U_2$ (coordinates (ζ, ξ)) in $\mathcal{U}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$ there exist functions $R \in \mathrm{RAT}^\zeta(\Omega)$ and $S \in \mathrm{RAT}^\xi(\Omega)$, neither identically zero, such that on Ω*

$$H_\gamma(\zeta, \xi) = \frac{R_\xi(\zeta, \xi)}{R(\zeta, \xi)} + \frac{S_\xi(\zeta, \xi)}{S(\zeta, \xi)}. \tag{4}$$

(3) *$\exists$ a point (ζ^*, ξ^*) and germs of meromorphic functions $R \in \mathrm{RAT}^\zeta_{(\zeta^*, \xi^*)}$ and $S \in \mathrm{RAT}^\xi_{(\zeta^*, \xi^*)}$, neither identically zero, such that about (ζ^*, ξ^*),*

$$\frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi) = \frac{\partial}{\partial \zeta} \left(\frac{R_\xi(\zeta, \xi)}{R(\zeta, \xi)} + \frac{S_\xi(\zeta, \xi)}{S(\zeta, \xi)} \right). \tag{5}$$

As motivation for the particular choice of ν, observe that for ξ suitably large,

$$H_\gamma(\zeta, \xi) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \left(\sum_{j=0}^{\infty} \frac{w^j}{\xi^{j+1}} \right) \frac{dz}{z - \zeta} = \sum_{j=0}^{\infty} \left(\frac{1}{2\pi \mathrm{i}} \int_\gamma w^j \frac{dz}{z - \zeta} \right) \frac{1}{\xi^{j+1}}. \tag{6}$$

Therefore, if γ bounds a holomorphic 1-chain T within $\mathbb{C}^2$, then one may see, by means of a residue calculation, that $H_\gamma(\zeta, \xi)$ constitutes a ξ-generating functions of the power sums of w-values (counting multiplicity, both positive and negative) of T over $z = \zeta$. In the more general case that γ bounds a holomorphic 1-chain within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$, a residue calculation of H_γ may exhibit some additional residue terms. The complete residue calculation is the basis of the following lemma.

Lemma 3.2. *Let γ be a closed 1-current with support contained in $\mathbb{C}^2$. Suppose γ bounds a holomorphic 1-chain V, with finite mass, within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Furthermore suppose that V has no components contained in $\{z_0 w_0 = 0\}$. Then there exists a meromorphic function $R(\zeta, \xi)$ being rational with respect to ζ on $\hat{\mathbb{C}} \times \mathcal{U}^{\xi}_{\gamma}$, a meromorphic function $S(\zeta, \xi)$ being rational with respect to ξ on $\mathcal{U}^{\zeta}_{\gamma} \times \hat{\mathbb{C}}$, and a rational function $T(\xi)$ on $\hat{\mathbb{C}}$, none of which are identically zero, such that*

$$H_{\gamma}(\zeta, \xi) = \frac{R_{\xi}(\zeta, \xi)}{R(\zeta, \xi)} + \frac{S_{\xi}(\zeta, \xi)}{S(\zeta, \xi)} - \frac{T_{\xi}(\xi)}{T(\xi)}, \tag{7}$$

on $\mathcal{U}^{\zeta}_{\gamma} \times \mathcal{U}^{\xi}_{\gamma}$.

Proof: (of Lemma 3.2) By separating the positive and negative components of V, we may express V as the difference of two positive holomorphic 1-chains, $V = V^+ - V^-$. Consider the individual boundaries $\gamma^+ = dV^+$ and $\gamma^- = dV^-$, which are closed and satisfy $\gamma = \gamma^+ - \gamma^-$. Hence, it suffices to consider the case where V is a positive holomorphic 1-chain.

Define $\mathcal{V}^{\zeta}_{V}$ to be the set of all ζ in $\mathbb{C}$ such that V has a component contained in $\{\zeta\} \times \hat{\mathbb{C}}$. Let V^{ζ} be the holomorphic chain V excluding all of its "vertical" components, i.e. those components contained in $\{\zeta\} \times \hat{\mathbb{C}}$ for some ζ. Locally each component of V^{ζ} is a branched analytic cover over a domain in $\mathcal{U}^{\gamma}_{\gamma}$ via the map π_1. Let $\mathcal{C}^{\zeta}_{V}$ denote the union of the critical points in $\mathcal{U}^{\zeta}_{\gamma}$ of all components of V^{ζ} when each component is viewed as an analytic cover via π_1. Define $\mathcal{I}^{\zeta}_{V} = \pi_1(\operatorname{spt} V^{\zeta} \cap \{w_0 = 0\})$.

If U_1' is a simply-connected domain in $\mathcal{U}^{\zeta}_{\gamma} \backslash (\mathcal{C}^{\zeta}_{V} \cup \mathcal{I}^{\zeta}_{V})$, then there exists a finite set of holomorphic functions $w_j(z)$ over U_1' whose graphs represent V^{ζ}, counting multiplicity. For ζ in U_1', let

$$S(\zeta, \xi) = \prod_{j=1}^{n} (\xi - w_j(\zeta)), \tag{8}$$

and note that

$$\frac{S_\xi(\zeta, \xi)}{S(\zeta, \xi)} = \sum_{j=1}^{n} \frac{1}{\xi - w_j(\zeta)}. \tag{9}$$

We may extend the definition of S to any connected component of $\mathcal{U}_\gamma^\zeta \times \hat{\mathbb{C}}$ by the equation

$$S(\zeta, \xi) = \xi^n - e_1(\zeta)\xi^{n-1} + \cdots + (-1)^n e_n(\zeta), \tag{10}$$

where each function $e_j(\zeta)$ is the jth elementary symmetric polynomial of the w-values of V^ζ over $z = \zeta$. The functions $e_j(\zeta)$ are holomorphically defined on $\mathcal{U}_\gamma^\zeta \backslash \mathcal{I}_V^\zeta$ and meromorphically defined on $\mathcal{U}_\gamma^\zeta$.

Consider the functions $e_j(\zeta)$ collectively near $\zeta = \infty$. Define M to be the largest order of a pole at ∞ among all the functions $e_j(\zeta)$, or define it to be 0 if none of the functions $e_j(\zeta)$ have a pole at ∞. Then $\frac{S(\zeta,\xi)}{\zeta^M}$ is polynomial in ξ with coefficients holomorphic in ζ at $\zeta = \infty$. Let $T(\xi) = \lim_{\zeta \to \infty} \frac{S(\zeta,\xi)}{\zeta^M}$. Observe that $\frac{T_\xi(\xi)}{T(\xi)} = \lim_{\zeta \to \infty} \frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}$.

Now we turn our attention to the other coordinate projection π_2, using definitions analogous to those used for π_1. Define $\mathcal{V}_V^\xi$ to be the set of all ξ in $\mathbb{C}$ such that V has a component contained in $\hat{\mathbb{C}} \times \{\xi\}$. Define V^ξ to be V with all its "horizontal" components removed. Let $\mathcal{C}_V^\xi$ denote the union of the critical points in $\mathcal{U}_\gamma^\xi$ for all components of V^ξ when each component is viewed as an analytic cover via π_2. Define $\mathcal{I}_V^\xi = \pi_2(\operatorname{spt} V^\xi \cap \{z_0 = 0\})$.

Let U_2' be a simply-connected domain in $\mathcal{U}_\gamma^\xi \backslash (\mathcal{C}_V^\xi \cup \mathcal{I}_V^\xi)$ and define a finite set of holomorphic functions $z_k(w)$ over U_2' whose graphs represent V^ξ, counting multiplicity. Then, for ξ in U_2', let

$$R(\zeta, \xi) = \left(\prod_{k=1}^{m} (\zeta - z_k(\xi)) \right)^{-1}, \tag{11}$$

and thus

$$\frac{R_\xi(\zeta,\xi)}{R(\zeta,\xi)} = -\sum_{j=1}^{n} \frac{z_k'(\xi)}{z_k(\xi) - \zeta}. \tag{12}$$

We may extend the definition of R to any connected component of $\hat{\mathbb{C}} \times \mathcal{U}_\gamma^\xi$ by the formula

$$R(\zeta, \xi) = \left(\sum_{k=0}^{m} (-1)^k f_k(\xi) \zeta^{m-k} \right)^{-1}, \tag{13}$$

where $f_k(\xi)$ is the kth elementary symmetric polynomial of the z-values of V^ξ over $w = \xi$.

Now we show that $H_\gamma(\zeta,\xi) = \frac{R_\xi(\zeta,\xi)}{R(\zeta,\xi)} + \frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)} - \frac{T_\xi(\xi)}{T(\xi)}$. It suffices to do this about a generic point $(\zeta^*, \xi^*) \in \left(\mathcal{U}_\gamma^\zeta \backslash (\mathcal{V}_V^\zeta \cup \mathcal{C}_V^\zeta \cup \mathcal{I}_V^\zeta)\right) \times \left(\mathcal{U}_\gamma^\xi \backslash (\mathcal{V}_V^\xi \cup \mathcal{C}_V^\xi \cup \mathcal{I}_V^\xi)\right)$. Since $dV = \gamma$, we may calculate H_γ by residues on V. The residues arising from the form $\nu = \frac{1}{\xi - w}\frac{dz}{z-\zeta} = \frac{w_0}{\xi w_0 - w_1}\left(\frac{z_0 dz_1 - z_1 dz_0}{z_0(z_1 - \zeta z_0)}\right)$ may be grouped into three classes: those arising at $z = \zeta$, those arising at $z = \infty$, and those arising at $w = \xi$. The cumulative total of the first class of residues is $\frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)}$ as relates to (9). The total of the second class of residues is $-\frac{T_\xi(\xi)}{T(\xi)}$. And the total of the third class of residues is $\frac{R_\xi(\zeta,\xi)}{R(\zeta,\xi)}$ as in the form of (12).

□

By the calculation in the proof of the previous lemma, one may note that if γ bounds a holomorphic 1-chain V within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ and if V avoids both the line

$\{z = \zeta^*\}$ and the line $\{w = \xi^*\}$, then $\frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi) = 0$ for (ζ, ξ) near (ζ^*, ξ^*). The converse is true and will later prove to be useful.

Lemma 3.3. *Let γ be a closed, rectifiable 1-current and suppose that* $\operatorname{spt} \gamma$ *is contained in $\mathbb{C}^2$ and satisfies condition A_1. Let $(\zeta^*, \xi^*) \in \mathfrak{U}^\zeta_\gamma \times \mathfrak{U}^\xi_\gamma$. The following are equivalent:*

(1) *γ bounds a holomorphic 1-chain, with finite mass, within $(\hat{\mathbb{C}} \backslash \{\zeta^*\}) \times (\hat{\mathbb{C}} \backslash \{\xi^*\})$.*

(2) *For (ζ, ξ) near (ζ^*, ξ^*),*

$$\frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi) = 0. \tag{14}$$

Proof: That condition 1 implies condition 2 is an immediate application of Lemma 3.2. Here we establish the reverse implication.

First we consider the case when $(\zeta^*, \xi^*) \in (\mathfrak{U}^\zeta_\gamma \times \mathfrak{U}^\xi_\gamma) \cap \mathbb{C}^2$. By calculating the series expansion of $\frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi)$ for (ζ, ξ) suitably near (ζ^*, ξ^*), one obtains that

$$\begin{aligned} \frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi) &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{1}{(\xi - \xi^*) - (w - \xi^*)} \frac{dz}{((z - \zeta^*) - (\zeta - \zeta^*)^2} \\ &= \frac{1}{2\pi \mathrm{i}} \int_\gamma \left(\sum_{j=0}^\infty \frac{-(\xi - \xi^*)^j}{(w - \xi^*)^{j+1}} \right) \left(\sum_{k=0}^\infty \frac{(k+1)(\zeta - \zeta^*)^k}{(z - \zeta^*)^{k+2}} \right) dz \\ &= \sum_{j=0}^\infty \sum_{k=0}^\infty \left(\frac{k+1}{2\pi \mathrm{i}} \int_\gamma \tilde{w}^{j+1} \tilde{z}^k \, d\tilde{z} \right) (\xi - \xi^*)^j (\zeta - \zeta^*)^k, \end{aligned} \tag{15}$$

where $\tilde{z} = \frac{1}{z - \zeta^*}$ and $\tilde{w} = \frac{1}{w - \xi^*}$, which serve as affine coordinates for $(\hat{\mathbb{C}} \backslash \{\zeta^*\}) \times (\hat{\mathbb{C}} \backslash \{\xi^*\})$. Condition 2 implies that $\int_\gamma \tilde{w}^j \tilde{z}^k d\tilde{z} = 0$ for $j \geq 1$ and $k \geq 0$. These are the monomial moments used by Wermer [14]. By way of polynomial approximation and integration by parts, their vanishing implies that γ satisfies the moment condition for $(\hat{\mathbb{C}} \backslash \{\zeta^*\}) \times (\hat{\mathbb{C}} \backslash \{\xi^*\})$. Thus γ bounds a holomorphic 1-chain within $(\hat{\mathbb{C}} \backslash \{\zeta^*\}) \times (\hat{\mathbb{C}} \backslash \{\xi^*\})$ [2].

In the special case that ζ^* or ξ^* equals ∞, a calculation similar to (15) with different coordinates, i.e. z instead of $\tilde{z}$ or w instead of $\tilde{w}$, leads to a similar conclusion. For instance in the case when $(\zeta^*, \xi^*) = (\infty, \infty)$, then for ζ and ξ suitably large,

$$(16) \quad \frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \left(\sum_{j=0}^{\infty} \frac{w^j}{\xi^{j+1}} \right) \left(\sum_{k=0}^{\infty} \frac{(k+1) z^k}{\zeta^{k+2}} \right) dz$$
$$= \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \left(\frac{k+1}{2\pi \mathrm{i}} \int_\gamma w^j z^k dz \right) \frac{1}{\xi^{j+1}} \frac{1}{\zeta^{k+2}}.$$

In the event that exactly one of the coordinates of the point (ζ^*, ξ^*) is infinite, then one can proceed with an appropriate hybrid of (15) and (16).

□

Proof: (of Theorem 3.1) Lemma 3.2 establishes the statement (1) $\implies$ (2). The implication (2) $\implies$ (3) is trivial. So it only remains to show (3) $\implies$ (1).

Let $R \in \mathrm{RAT}^{\zeta}_{(\zeta^*, \xi^*)}$ and $S \in \mathrm{RAT}^{\xi}_{(\zeta^*, \xi^*)}$ satisfy condition 3. Let U_1 and U_2 be simply-connected proper domains of $\hat{\mathbb{C}}$ such that $U_1 \times U_2$ is a neighborhood of (ζ^*, ξ^*) on which R and S are defined (as meromorphic functions). Define monic polynomials $P_R, Q_R \in \mathcal{M}(U_2)[\zeta] \backslash \{0\}$ and a function $a_R(\xi) \in \mathcal{M}(U_2)$ such that $R(\zeta, \xi) = a_R(\xi) \frac{P_R(\zeta, \xi)}{Q_R(\zeta, \xi)}$. Similarly define monic polynomials $P_S, Q_S \in \mathcal{M}(U_1)[\xi] \backslash \{0\}$ and a function $a_S(\zeta) \in \mathcal{M}(U_1)$ such that $S(\zeta, \xi) = a_S(\zeta) \frac{P_S(\zeta, \xi)}{Q_S(\zeta, \xi)}$. Let V_R be the zero divisor of P_R minus the zero divisor of Q_R in $\mathbb{C} \times U_2$. Similarly let V_S be the zero divisor of P_S minus the zero divisor of Q_S in $U_1 \times \mathbb{C}$. It holds that V_R is the divisor of R with horizontal components removed, and that V_S is the divisor of S with vertical components removed.

Let Δ_1 be a (bounded) disk with closure contained in $U_1\backslash(\mathcal{C}^{\zeta}_{V_S}\cup\mathcal{I}^{\zeta}_{V_S})$, using the definition of $\mathcal{C}^{\zeta}_{V}$ and $\mathcal{I}^{\zeta}_{V}$ given in the proof of Lemma 3.2. Similarly let Δ_2 be a disk with closure contained in $U_2\backslash(\mathcal{C}^{\xi}_{V_R}\cup\mathcal{I}^{\xi}_{V_R})$. Shrinking Δ_1 and Δ_2, if necessary, we may suppose that V_R and V_S avoid $\Delta_1\times\Delta_2$ and that a_R and a_S are holomorphic and non-vanishing over Δ_2 and Δ_1, respectively. Redefine (ζ^*,ξ^*) to be some point in the polydisk $\Delta_1\times\Delta_2$.

Let $W_S = V_S\llcorner(\Delta_1\times\hat{\mathbb{C}})$ and $W_R = V_R\llcorner(\hat{\mathbb{C}}\times\Delta_2)$. Define their boundaries $\Gamma_S = dW_S$ and $\Gamma_R = dW_R$. So Γ_S and Γ_R are finite 1-chains contained in $\mathbb{C}^2$, whose components each have finite length and bound an analytic disk within $\mathbb{C}^2$. So the support of $\Gamma_S-\Gamma_R$ satisfies condition A_1, and $\Gamma_S-\Gamma_R$ bounds a finite holomorphic 1-chain W_S-W_R within $\hat{\mathbb{C}}\times\hat{\mathbb{C}}$.

By tracing the residue calculation in the proof of Lemma 3.2, one may see that $H_{\Gamma_S-\Gamma_R}(\zeta,\xi)=\frac{\partial}{\partial\xi}\log(\frac{P_S(\zeta,\xi)}{Q_S(\zeta,\xi)}\frac{P_R(\zeta,\xi)}{Q_R(\zeta,\xi)})$ for $(\zeta,\xi)\in\Delta_1\times\Delta_2$. Let $\Gamma=\gamma-\Gamma_S+\Gamma_R$ and observe that

$$\frac{\partial}{\partial\zeta}H_\Gamma(\zeta,\xi)=\frac{\partial}{\partial\zeta}\left(H_\gamma(\zeta,\xi)-H_{\Gamma_S-\Gamma_R}(\zeta,\xi)\right)=\frac{\partial^2}{\partial\zeta\partial\xi}\log(a_R(\xi)a_S(\zeta))=0,$$

for $(\zeta,\xi)\in\Delta_1\times\Delta_2$. By Lemma 3.3, Γ bounds an holomorphic 1-chain W within $(\hat{\mathbb{C}}\backslash\{\zeta^*\})\times(\hat{\mathbb{C}}\backslash\{\xi^*\})$. The current $W+W_S-W_R$ constitutes a holomorphic 1-chain in $(\hat{\mathbb{C}}\times\hat{\mathbb{C}})\backslash\operatorname{spt}\gamma$, which follows by the Harvey-Shiffman Theorem [10], [1]. So γ bounds a holomorphic 1-chain, namely $W+W_S-W_R$, within $\hat{\mathbb{C}}\times\hat{\mathbb{C}}$.

□

We see that both of the parameterized forms ν and ω can be employed to characterize the boundaries of holomorphic 1-chains within $\hat{\mathbb{C}}\times\hat{\mathbb{C}}$ (and $\mathbb{CP}^2$). As pointed out in (15) and (16), $H_\gamma(\zeta,\xi)$ may be viewed as a generating function of the Wermer monomial moments. Similarly $\int_\gamma\omega$ can be viewed as a generating function of these moments, though with differing constants and a differing arrangement of coefficients. For example, we have that

$$(17) \quad \frac{1}{2\pi \mathrm{i}} \int_\gamma z_1 \frac{d(z_2 - \eta z_1)}{z_2 - \xi - \eta z_1}$$
$$= \sum_{j=0}^{\infty} \frac{-1}{2\pi \mathrm{i}} \left(\int_\gamma z_1 (z_2 - \eta z_1)^j \, d(z_2 - \eta z_1) \right) \frac{1}{\xi^{j+1}} = \sum_{j=0}^{\infty} \frac{1}{2\pi \mathrm{i}} \left(\int_\gamma \frac{(z_2 - \eta z_1)^{j+1}}{j+1} \, dz_1 \right) \frac{1}{\xi^{j+1}}$$
$$= \sum_{j=0}^{\infty} \sum_{k=0}^{j+1} \left(\frac{\binom{j+1}{k}}{j+1} \frac{(-1)^k}{2\pi \mathrm{i}} \int_\gamma z_1^k z_2^{j+1-k} \, dz_1 \right) \eta^k \frac{1}{\xi^{j+1}},$$

for ξ large and η bounded.

So γ bounds within $\mathbb{C}^2$ if and only if the Wermer moments are zero, which is equivalent to the identical vanishing of $\int_\gamma \nu$ or $\int_\gamma \omega$ for some appropriate region of parameters. In contrast the condition that γ bounds within $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$ is equivalent to the existence of some relationship among the Wermer moments of γ that can be encapsulated in statements using either $\int_\gamma \nu$ or $\int_\gamma \omega$ as generating functions. (It is possible that other generating functions, besides these two, may also be worth studying.) One benefit of considering H_γ is that it naturally leads to a characterization of the boundaries of holomorphic 1-chains within $\mathbb{C} \times \hat{\mathbb{C}}$, which is the topic of the next section.

4. Characterizations within $\mathbb{C} \times \hat{\mathbb{C}}$

By the framework established in Theorem 3.1 and Lemma 3.2, we produce the following characterization within $\mathbb{C} \times \hat{\mathbb{C}}$.

Theorem 4.1. *Let γ be a closed, rectifiable 1-current and suppose that* $\operatorname{spt} \gamma$ *is contained in $\mathbb{C}^2$ and satisfies condition A_1. The following are equivalent:*

(1) *γ bounds a holomorphic 1-chain, with finite mass, within $\mathbb{C} \times \hat{\mathbb{C}}$*

(2) *For any* (ζ^*, ξ^*) *with any connected neighborhood* $\Omega = U_1 \times U_2$ *(coordinates* (ζ, ξ)*) in* $\mathfrak{U}_\gamma^\zeta \times \mathfrak{U}_\gamma^\xi$ *with* ζ^* *in the component of* $\mathfrak{U}_\gamma^\zeta$ *containing* ∞*, there exists a function* $R \in \mathrm{RAT}^\zeta(\Omega) \cap \mathcal{O}^*(\Omega)$*, such that on* Ω

$$H_\gamma(\zeta, \xi) = \frac{R_\xi(\zeta, \xi)}{R(\zeta, \xi)}. \tag{18}$$

(3) $\exists$ (ζ^*, ξ^*) *with* ζ^* *in the component of* $\mathfrak{U}_\gamma^\zeta$ *containing* ∞*, and a germ of a meromorphic function* $R \in \mathrm{RAT}^\zeta_{(\zeta^*, \xi^*)}$*, not identically zero, such that about* (ζ^*, ξ^*)*,*

$$\frac{\partial}{\partial \zeta} H_\gamma(\zeta, \xi) = \frac{\partial}{\partial \zeta}\left(\frac{R_\xi(\zeta, \xi)}{R(\zeta, \xi)}\right). \tag{19}$$

Proof: Let U_1' denote the component of $\mathfrak{U}_\gamma^\zeta$ containing ∞. Assume that there is a holomorphic 1-chain V, with finite mass, bounded by γ within $\mathbb{C} \times \hat{\mathbb{C}}$. By the maximum principle it holds that $\operatorname{spt} V \cap (U_1' \times \hat{\mathbb{C}}) = \emptyset$. Therefore proceeding via the calculation in Lemma 3.2 with $\Omega \subseteq U_1' \times \hat{\mathbb{C}}$, we observe that $\frac{S_\xi(\zeta,\xi)}{S(\zeta,\xi)} = 0$ on Ω, and that $\frac{T_\xi(\xi)}{T(\xi)} = 0$. So (18) holds on Ω. Since H_γ is holomorphic on $\mathfrak{U}_\gamma^\zeta \times \mathfrak{U}_\gamma^\xi$, R cannot have a pole or a zero on Ω. Thus 1 implies 2.

Condition 2 clearly implies 3, so it only remains to show that 3 implies 1. So assume condition 3 which implies condition 3 of Theorem 3.1 with $S = 1$ and R as given. Let V be the holomorphic 1-chain, with finite mass, bounded by γ as constructed in the proof of Theorem 3.1. Owing to the construction of V using $S = 1$, plus a maximum value argument, it follows that $\operatorname{spt} V \subseteq (\hat{\mathbb{C}} \backslash U_1') \times \hat{\mathbb{C}} \Subset \mathbb{C} \times \hat{\mathbb{C}}$. So V is bounded by γ within $\mathbb{C} \times \hat{\mathbb{C}}$.

□

By inverting the logarithmic derivative we may express this characterization in the following way.

Corollary 4.2. *Let γ be a closed, rectifiable 1-current and suppose that* $\operatorname{spt}\gamma$ *is contained in $\mathbb{C}^2$ and satisfies condition A_1. Let $(\zeta^*, \xi^*) \in \mathcal{U}^\zeta_\gamma \times \mathcal{U}^\xi_\gamma$, with ζ^* in the component of $\mathcal{U}^\zeta_\gamma$ containing ∞. Let U_1 and U_2 be connected neighborhoods of ζ^* and ξ^* contained in $\mathcal{U}^\zeta_\gamma$ and $\mathcal{U}^\xi_\gamma$, respectively. Suppose that U_2 is simply-connected and let $G(\zeta, \xi) = \exp(\int_{\xi^*}^{\xi} H_\gamma(\zeta, \psi)\, d\psi)$ on $U_1 \times U_2$, defining the definite integral by using any path between ξ^* and ξ in U_2. The following are equivalent:*

1. *γ bounds a holomorphic 1-chain, with finite mass, within $\mathbb{C} \times \hat{\mathbb{C}}$*
2. *$G \in \mathrm{RAT}^\zeta(U_1 \times U_2) \cap \mathcal{O}^*(U_1 \times U_2)$.*

Proof: Observe that $H_\gamma(\zeta, \xi) = \frac{\partial}{\partial \xi}(\log G(\zeta, \xi)) = \frac{G_\xi(\zeta,\xi)}{G(\zeta,\xi)}$. Thus 2 implies 1 using Theorem 4.1. Conversely, if γ bounds a holomorphic 1-chain, with finite mass, within $\hat{\mathbb{C}} \times \mathbb{C}$, then, by Corollary 4.2, there exists an $R \in \mathrm{RAT}^\zeta(U_1 \times U_2) \cap \mathcal{O}^*(U_1 \times U_2)$ satisfying (18) on $U_1 \times U_2$. It follows from (18) and the definition of G that $G(\zeta, \xi) = \frac{R(\zeta,\xi)}{R(\zeta,\xi^*)}$. So $G \in \mathrm{RAT}^\zeta(U_1 \times U_2) \cap \mathcal{O}^*(U_1 \times U_2)$.

□

5. A Connection to the Cauchy Integral

The integral H_γ bears a structural similarity to the standard Cauchy integral. But we will also point out a connection between the role of H_γ presented in this article and the role of the Cauchy integral in characterizing the boundary values of meromorphic and holomorphic functions.

Let U be a domain in $\mathbb{C}$ with $\mathcal{C}^1$ boundary and let f be a Hölder continuous function on ∂U. The function f is the boundary value of a holomorphic function on U if and only if its Cauchy integral $\frac{1}{2\pi \mathrm{i}} \int_{\partial U} f(z) \frac{dz}{z-\zeta}$ vanishes for ζ outside $\overline{U}$, [11](page 113). The function f is the boundary value of a meromorphic function if and only if its Cauchy integral is a rational function for ζ outside $\overline{U}$. (The forward implication follows by way of a residue calculation. For the converse, suppose $\rho(\zeta)$ is a rational function that agrees with the Cauchy integral for ζ outside of U. Then $\frac{1}{2\pi \mathrm{i}} \int_{\partial U} f(z) \frac{dz}{z-\zeta} - \rho(\zeta)$ is a meromorphic extension of f to U, which follows by Plemelj's Theorem [13].)

Now let us consider γ to be given by the graph of f over ∂U, with orientation such that $(\pi_1)_* \gamma$ bounds U positively. The curve γ bounds a holomorphic 1-chain within $\mathbb{C} \times \hat{\mathbb{C}}$ if and only if it bounds an analytic variety V in $U \times \hat{\mathbb{C}}$ that is one-sheeted over U, which is equivalent to f having a meromorphic extension to U. Similarly γ bounds within $\mathbb{C}^2$ if and only if f has a holomorphic extension to U.

In the above context, the Cauchy integral of f can be simply expressed as $\frac{1}{2\pi \mathrm{i}} \int_\gamma w \frac{dz}{z-\zeta}$. When γ is not confined to the special case above, then γ may bound a holomorphic 1-chain that is multi-sheeted over portions of $\mathbb{C}$, and the integral $\frac{1}{2\pi \mathrm{i}} \int_\gamma w \frac{dz}{z-\zeta}$ gives the sum, counting multiplicity, of the w-values for each fiber above $z = \zeta$. Thinking combinatorially, this by itself does not provide enough data to reconstruct each fiber. But a finite set of values w_m can be determined by their power sums. As noted earlier in connection to (6) and the motivation for the construction of ν, the fiber-wise sums of powers of w-values correspond to the integrals $\frac{1}{2\pi \mathrm{i}} \int_\gamma w^j \frac{dz}{z-\zeta}$, and H_γ is a generating function of these integrals. So from this vantage point, Theorem 4.1 and Lemma 3.3 (with $\zeta^* = \infty$, $\xi^* = \infty$) may also be viewed as natural multi-sheeted generalizations of the Cauchy integral method for characterizing the boundary values of meromorphic and holomorphic functions.

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